How Might Computer Algebra Change Senior Mathematics: The Case of Trigonometry

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Biographical details

Kaye Stacey is Foundation Professor of Mathematics Education at the University of Melbourne. She is a researcher, teacher educator and a supervisor of some wonderful research students; she is a well-known author of books and articles for mathematics teachers and of research papers. Lynda Ball was until recently Mathematics co-ordinator and senior Mathematics teacher at Kew High School in Melbourne. Lynda has now joined Kaye at the University working with the CAS-CAT project team, which is looking at potential changes in senior mathematics when computer algebra systems are readily available to all senior school mathematics students.

Synopsis

Computer algebra systems (CAS) have been available on computers for many years. However, they will soon be available on affordable hand-held machines, and at that stage, are poised to make an impact on school mathematics curriculum and assessment. This paper will demonstrate how the teaching of trigonometry and circular functions may change. Good basic algebra will remain essential, the order of topics may change, and some topics will no longer be justifiable whilst others will be tackled in new ways. The interplay between graphs and symbols can be strengthened.

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Purpose

Computer algebra systems have been available for use on computers now for many years and it has long been common for tertiary mathematics and engineering courses to make some use of them. They have not, however, made a large impact on common practice and their impact in schools is minimal. This may change in the near future, when the capability of graphics calculators is extended to include a computer algebra system (CAS) at a price within reach of most Australian senior mathematics students. When a technology is regularly available at the time and place where mathematics is done, it can make a significant difference to practice.

This paper reports preliminary thinking on how one of the topics in Australian senior mathematics courses (trigonometry) may be impacted by readily available CAS. It has been prepared as part of the preliminary thinking for conducting an experimental Year 12 mathematics subject using CAS in Victoria. Further details of this, the CAS-CAT project, are available from http://www.edfac.unimelb.edu.au/DSME/CAS-CAT/. This paper aims to provide a general discussion of how various topics related to trigonometry and circular functions might be taught with CAS. It is not confined to one current syllabus, but looks at a range of topics that may be included at senior secondary level. Sample questions were mainly derived by examining textbooks such as Evans et al (1999) and Fitpatrick et al (1992). The paper outlines some of the new approaches that should be considered for adoption in a CAS environment, proposes topics to receive different, more, earlier or less emphasis and points out some of the differences between systems that will need to be considered for assessment. A range of examples is considered to illustrate particular points.

In preparation for this paper, we have used several different systems: three hand-held calculators (TI89 manufactured by Texas Instruments, the Casio FX2.0, the Hewlett-Packard HP49G) and two computer packages Derive and Mathematica. As might be expected with emerging and complicated software such as a CAS, there are currently substantial differences between the modes of use and the form of the answers obtained (which are important for setting examinations) although there is a broad set of common features.

Trigonometry and circular functions are excellent topics to remain in a CAS-active curriculum because of the very accessible and important fields of applications; finding lengths and angles in two and three dimensions and the modelling of periodic phenomena. There is also important theoretical work because the properties of the family of trigonometric functions (sine, cosine and tangent) are accessible by elementary techniques including calculus but they contrast with properties of polynomial and exponential/logarithmic functions. For this paper, senior secondary work with trigonometric functions is divided into three main areas treated in turn:

firstly solving triangles, secondly working with identities, and thirdly the function properties.

1. Solving triangles

Numerical or exact values

The advent of the scientific calculator revolutionised the arithmetic involved in solving triangles and made the use of tables of values of the functions obsolete. Computer algebra extends this revolution in two ways, by using exact values and by performing some of the equation solving arising in multi-constraint problems. Finding an 'opposite' side in a triangle given an angle of 60 degrees and a hypotenuse 12 units long is straightforward if the calculator is set in numerical mode and to use degrees: 12 * sin 60 = 10.39. However, in exact mode, the CAS gives $6\sqrt{3}$. In an examination system with CAS, no longer would a question such as this test any special knowledge of the values of the sine function. The role of work with exact values needs to be carefully reconsidered: it could be a major feature, supported by technology, but there must be a clearly specified purpose.

The computer algebra systems can also provide exact values beyond the commonly memorised repertoire. For example, entering cos(75°) on the FX2.0 with exact mode set will give the $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ in one step. Mathematica gives $\frac{-1+\sqrt{3}}{2\sqrt{2}}$. This variety illustrates that students will frequently need basic algebra skills, if they are to match output to given forms (e.g. for checking answers in a textbook, working with others using a different system, getting a specified answer on a test etc).

As with a scientific calculator, students need to be aware of degrees and radians as two parallel systems for measuring angles. However, radian measure is more important on a CAS than for a scientific calculator because it is used in more advanced features such as the calculus and even in some equation solving.

The equation solving facility can also be used with advantage. Kendal and Stacey (1996) studied the trigonometry problem solving of all the Year 10 students at a large Victorian high school. They found that the most frequent cause of errors in one-step problems to find an unknown side in a right angle triangle was not the trigonometry but the basic algebra, especially when the unknown side is in the denominator of the trigonometric ratio. CAS can, however, solve an equation such as 100/x = tan (15°) immediately. For example, the TI89 in exact mode gives the exact answer $\frac{100}{\sqrt{3}+2}$. A numerical answer is also immediate.

A word of caution is due here. Kendal and Stacey also found that students who encountered equations such as $30=\frac{0.0003}{x}$ in the course of learning trigonometry made significant improvements on solving algebraic equations like this. In schools such as the one where Kendal and Stacey collected their data, trigonometry is a vehicle for improving basic algebra, although progressing in the new topic is also made more difficult by the students' lack of skill. In considering changes to in one topic such as trigonometry, the indirect roles in the curriculum need to be considered, as well as the direct role.

Introducing new methods into problem solving

Solving triangles in multiple-step problems can often be done using the facility to solve linear equations. In these questions, students will be able to think through the whole process to solve the problem, rather than just what operation is required at each stage to give the next line in the solution. Consider this problem:

> A flagpole of height h metres is on the top of a tower of height H metres. From a point horizontally 85 metres away from the base, the angle of elevation is 40˚ to the top of tower and 43˚ to the top of the flagpole. How tall is the flagpole?

Writing down the information from the triangles:

$$\begin{aligned} H &= 85 \tan 40^{\circ} \\ H + h &= 85 \tan 43^{\circ} \end{aligned}$$

These equations can be solved as

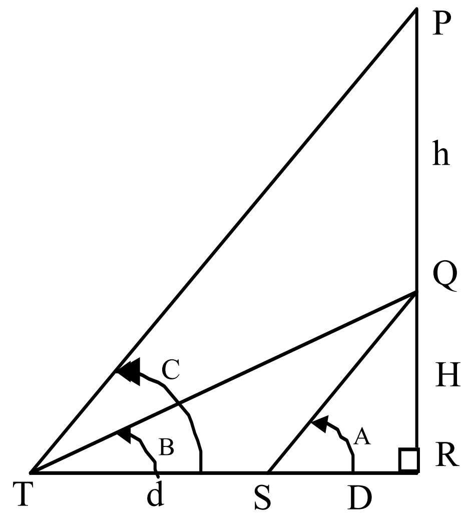
$$h = 85 \tan 43^{\circ} - h = 85 \tan 40^{\circ}$$

In this problem, student understanding is evident through the setting up of the problem. The mathematical formulation of the problem is essential in order to be able to use the CAS to find the numerical solution.

If these equations were a little harder, we could see them as two linear equations in two unknowns and use the matrix solving facilities – this is a general method that could be used in a wide range of trigonometric situations. For example, consider the problem in Figure 1.

$$\begin{array}{lllll} H + h & = (35 + D)\tan 48^{\circ} & & H + h - \tan 48^{\circ} D & = 35.\tan 48^{\circ} \\ H & = D \tan 43^{\circ} & \text{rearranged as} & H - D \tan 43^{\circ} D & = 0 \\ H & = (35 + D)\tan 40^{\circ} & & H - \tan 40^{\circ} D & = 35.\tan 40^{\circ} \end{array}$$

After rearranging these as three linear equations in three unknowns (H, h and D) as shown, the matrix facility can be used to solve them automatically. This emphasises an orientation to writing down the whole problem and then using an automated procedure for the solution, rather than working step-by-step through a solution. It also shows new links between previously isolated topics. A discussion of this problem in the case where A, B, C and D are unknown, and the symbolic algebra facility is used, is given by Stacey (1999).



A flagpole (PQ in the diagram) is placed on top of a castle wall (QR), which is surrounded by a moat (RS). From point S, the angle of elevation of the top of the wall is A = 43˚. From point T, the angle of elevation of the top of the castle wall is B = 40˚. From point T, the angle of elevation of the top of the flagpole is C = 48˚ The distance ST is 35 metres. Find the height of the flagpole.

Figure 1. An illustration that matrix equation solving can be used in trigonometry problems.

Opportunities for more 'algebraic' approaches

There are other opportunities for more 'algebraic' approaches when using CAS e.g. less step by step evaluation, more focus on writing down the constraints for a problem, more opportunities to work with unspecified values. Students will be able to show how a complete expression can be made, rather than just working everything out step-by-step and evaluating at each stage. The reasons why such an approach may be considered more algebraic comes from a consideration of the differences between arithmetic and algebraic thinking, as proposed, for example, by Stacey and MacGregor (2000).

Example:

> 'Two ports, *A* and *B* are such that *B* is due West of *A*. *A* is due North of a ship, *S*. The ship is on a course 328°T and reaches *B* after travelling for 3 hrs at 25 km/h. Calculate the distance between the two ports, and the time it would have taken the ship to reach *A* from *S*.' (Fitzpatrick, Galbraith and Henry, 1992, p. 237).

Outline of a solution using CAS:

Define AB = 75.sin (360°–328°)

Using Pythagoras' theorem, the distance from A to S is $D = \sqrt{75^2 - AB^2}$

Define the time from A to S to be equal to D/25

Then the whole expression can be evaluated at the end in one step to get 2.54 hours

We propose that the written record from a student using CAS to assist in solving this problem might look something like the outline above and welcome debate on this point. (A discussion paper on this issue is available on the CAS-CAT web-site, address above.)

2. Trigonometric identities

Beyond simple relationships, working with trigonometric identities on CAS calculators can be quite difficult and there are substantial variations between brands. This aspect of CAS may well become easier to use in the near future, but with the current capability, it often seems that substantial by-hand skills are required for a user to persuade the CAS to work with trigonometric identities.

Well known relationships between cos, sin and tan

Relations such as sin (90 – x) = cos(x), sin(180 – x) = sin x, cos(π – x) = – cos x, tan(π – x) = –tan x etc are mostly easy to find on all CAS machines. There is little purpose for these identities in calculation now and so we propose that the emphasis should be on how these translate into properties of the functions, rather than for calculation. Students should be able to relate these identities to the various symmetries of the functions and link the algebraic with the graphical. These identities are particular instances of the compound angle formulae such as sin(A + B)= sin A cos B + cos A sin B, which the systems use frequently. For this reason, students may need to learn about the existence of these relationships earlier than before to make sense of some output.

Identities can be surprisingly hard

Simplifying an expression such as [(cos t/tan t– sin t tan t) sin t cos t]/(cos t – sin t) demonstrates the variability between currently available computer algebra systems. Mathematica will simplify the expression in one step to ($\frac{1}{2}$)(2 + sin(2t)). On the hand-held calculators, the user needs to enter parts of the expression separately before combining them. The challenge for students is in deciding when it is appropriate to collect trigonometric terms versus expanding, and how much of the expression should be entered at each stage. This shows that although it may not be possible to simplify an expression in one step, through using inbuilt features of the calculator it may still be possible to simplify the expression. Students need to use their understanding of trigonometry to decide on how they can work towards a simplified expression, but problems like this may well be easier by hand.

Easy by hand, harder by CAS

Solving equations involving trigonometric functions can also be easier by hand than with the machine. For example, it is quite easy to solve the equation $5\ \cos^2 x + 2\ \sin^2 x = 2$ by hand because the substitution of $\cos^2 x + \sin^2 x = 1$ is immediately recognisable. The CAS machines, however, gave a variety of answers, sometimes after substantial guidance from the user. The HP 49G solved the equation numerically immediately giving one answer of 1.5707, but solving the equation exactly had to be guided step by step by the user. Using the solve command on the TI 89 gives the exact answer x=((2n – 1)π)/2. The FX 2.0 gives an answer of x=(2π)k – π/2 and x=(2π)k + π/2. Even though the systems give general solutions for the problem, the forms of the answers are different and students must be able to interpret these different forms. Using Mathematica only two answers were obtained; $x=-\pi/2$ and $x=\pi/2$.

This highlights the differing capabilities of various brands of calculators (and CAS software packages) for given content areas. The trigonometric features on the various brands of CAS calculators seem quite different. Students will need to know a range of trigonometric relationships to make sensible decisions when using the trigonometry menus.

3. Properties of trigonometric functions

The properties of trigonometric functions provide an excellent contrast in the curriculum to the properties of polynomials and exponential/logarithmic functions. Students learn how to manipulate the functions to match given amplitudes and periods for modelling and to deal with multiple solutions in a regular pattern. When using CAS calculators, all the graphing capabilities of graphics calculators are retained, so that many of the adjustments required have already been widely discussed and are in the process of being implemented in some examination systems. Students can plot graphs readily, read off periods and amplitudes and maxima and minima etc, see families of multiple solutions graphically, and solve equations graphically.

Students will need to know more about parameters

To use an algebraic language such as CAS, students need to know of many different uses of algebraic letters, including as parameters not just as unknowns and variables. For example, to find all values of x such that $\sin x = \frac{-\sqrt{3}}{2}$ in exact mode, the HP 49G gives $x = -\frac{(6n1-4)\pi}{3}$. To find solutions that satisfy specified constraints such as $0 \le x \le 2\pi$, students must deal with the parameters.

Using Mathematica, the only solution that is obtained to this equation is $-\pi/3$, although the program reported that some solutions might not have been found. Students would need to recognise that the expected form of the solution is a family of solutions and use this understanding to generate the correct solution for the problem.

Finding all families of solutions is a delicate operation and students using CAS will need to be very careful not to rely on it mindlessly. The following investigation, illustrated in Figure 2, could be used to show some of the difficulty of dealing with functions which are not one to one and the consequences of defining inverses by limiting the domains.

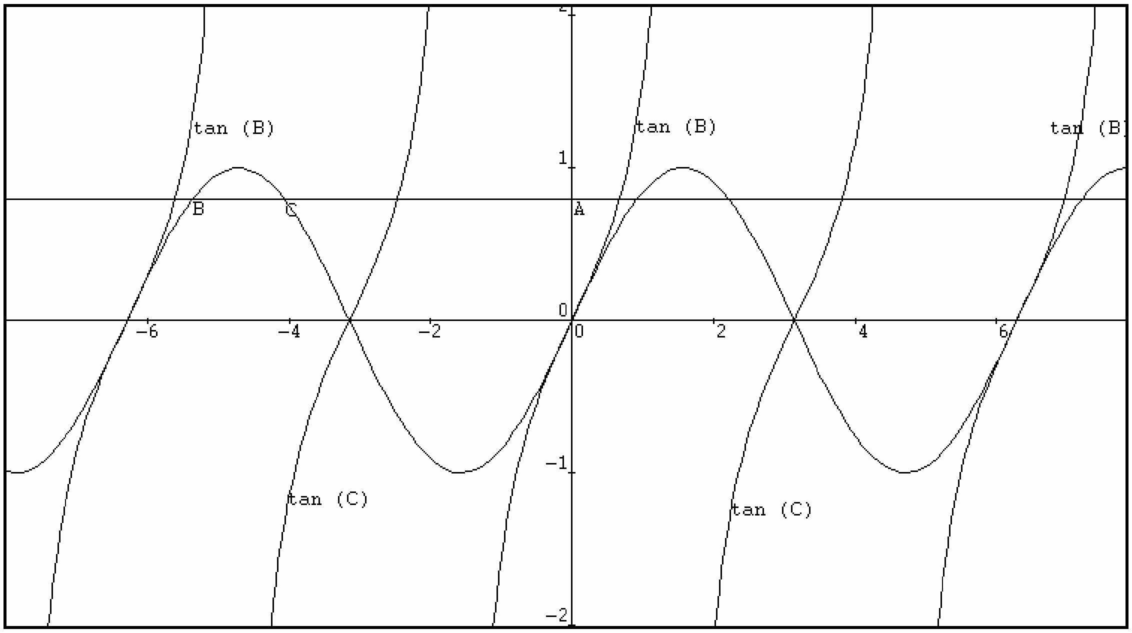


Figure 2. Investigation. If sin x = A, what is the value of tan x?

Figure 2 shows the graphs of y = sin x and y = tan x. The horizontal line y = A intersects the graph where sin x = A (an arbitrary value). Two such points, B and C, are labelled. The problem asks us to find the value of the tangent function at points like these. Moving up from point B gives tan(B), marked on the graph and moving down from point C gives tan (C). The graph shows that there are a series of points where the value will be tan (B), spaced at intervals of 2π and a series of points where the value will be tan (C), also spaced at intervals of 2π.

What does the algebra facility do? Quick calculation of tan(a sin A) gives one solution $A/\sqrt{(-(A^2 - 1)}$ immediately, for example using the HP 49G, but this is only one of the two solutions (in the case shown in Figure 2, it is tan (B)) and there is no indication that there are families of solutions. The relationship between tan (B) and tan (C) can be found by considering the symmetries of the graphs.

Directions for doing trigonometry with CAS

The examples above have discussed some of the ways in which the trigonometry topics in senior secondary school may change when CAS is widely available. The strengths (and hence challenges to the mathematics curriculum as it is) of the current systems over the graphics calculators lies in their ability to deal with exact solutions and the equation solving (both numeric and exact), Some identities which are now memorised, such as sin(−x) = sin(x), are available at the push of a button, and compound angle formulas are readily used. On the other hand, the skill involved in persuading the machine to solve some trigonometric identities surpasses the skill required to do them by hand.

Several topics might be introduced earlier to students using CAS. The existence of radians must be known early, at least to set the MODE to degrees. Because the CAS needs to work in radians in unexpected places, radians cannot be ignored as easily as on a graphics or scientific calculator.) Compound angle formulas may be used by the CAS in unexpected places for simplification and so at least their existence needs to be known early. (For example, the HP 49G frequently replaces $\cos^2 x$ by $\frac{1}{2}(\cos 2x + 1)$ unexpectedly to me). However, since facilities like this are so strong, possibly students need to learn little other than their existence. This is a specific instance of the question of what essential skills (or understandings) students will need to be able to use the trigonometric functions effectively. For example, will students need to know the different forms of cos(2x) etc so that they can make sensible decisions when using the trig menus? Students will also need to be able to work with parameters. Finally, it is time to bid goodbye to the cosec, cot and sec functions. The CAS manages without them, and we think we can too.

In this topic as with others, CAS provides opportunities for what can be seen as a more 'mathematical' approach to solving problems; one where there is less step by step evaluation, but more focus on writing down the constraints for a problem, and more opportunities to work with unspecified values. This may help students to display in their written work their overall plan and the reasons for it, rather than principally recording the details of the calculations.

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