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THE IMPACT OF TEACHER PRIVILEGING ON LEARNING DIFFERENTIATION WITH TECHNOLOGY

ABSTRACT. This study examines how two teachers taught differentiation using a hand held computer algebra system, which made numerical, graphical and symbolic representations of the derivative readily available. The teachers planned the lessons together but taught their Year 11 classes in very different ways. They had fundamentally different conceptions of mathematics with associated teaching practices, innate ″privileging″ of representations, and of technology use. This study links these instructional differences to the different differentiation competencies that the classes acquired. Students of the teacher who privileged conceptual understanding and student construction of meaning were more able to interpret derivatives. Students of the teacher who privileged performance of routines made better use of the CAS for solving routine problems. Comparison of the results with an earlier study showed that although each teacher's teaching approach was stable over two years, each used technology differently with further experience of CAS. The teacher who stressed understanding moved away from using CAS, whilst the teacher who stressed rules, adopted it more. The study highlights that within similar overall attainment on student tests, there can be substantial variations of what students know. New technologies provide more approaches to teaching and so greater variations between teaching and the consequent learning may become evident.

KEYWORDS: calculus, computer algebra systems (CAS), symbolic calculators, multiple representations of derivative, teacher privileging, classroom teaching practices.

INTRODUCTION

The recent availability of Computer Algebra Systems (CAS) offers teachers the opportunity to make mathematics accessible to a wider range of students, and students the chance to develop a more holistic understanding of fundamental concepts. Of special relevance to the teaching of calculus is the facility of computer algebra systems to support and link numerical, graphical and symbolic representations of function that authors such as Dick (1996), Hillel (1993), and Tall (1996) suggest should enhance conceptual understanding. Many studies have made claims that using a CAS improves understanding of calculus concepts (e.g., Heid, 1988; Palmiter, 1991; & Repo, 1994; and others) but this is not always the case (Penglase & Arnold, 1996). For example, Ellison (1994) found that while most students improved their conceptual understanding (in a calculus course that focused on multiple representations and links between them) some students developed only partial conceptual understanding. Reasons for this lack of success cannot be determined due to a lack of information about the ″context″ of the teaching.

As observed by Penglase and Arnold (1996), for resolution of this issue, it is essential to distinguish the use of the tool from the effect of teaching. The present study aims to explore the impact of using a CAS on understanding calculus concepts in the context of a detailed analysis of the teaching process.

Computer algebra systems may change many methods of problem solving from being methods ″in principle″ to being methods ″in practice″. Whereas symbolic methods using standard rules have been overwhelmingly the most practical methods of differentiation without technology, with a CAS, graphical and numerical methods which were previously impractical can now be quick and easy. For example, quite short sequences of button presses using a modern hand-held CAS such as the Texas Instruments calculator TI-92 will (usually) find the derivative of a function at a point by any of the following methods:

- symbolic differentiation followed by substitution,
- finding the value of a limit expression using symbolic manipulation,
- drawing a graph and the tangent at a point (an automated process) and deducing the gradient of the tangent from its equation,
- drawing a graph and some nearby secants from which the limit of their gradients can be guessed,
- calculating difference quotients using nearby values of the function and estimating the limit.

The growth in options for solving problems is accompanied by a growth in options for teaching. This study sets out to observe how two teachers dealt with these widened options as they taught introductory differential calculus and how their choices (e.g., of how to explain ideas, representations to stress, and degree of emphasis on by-hand techniques) were reflected in what their students learned.

The present study arose from an exploratory study where we analysed the ways in which three teachers taught a twenty lesson unit on introductory differential calculus to Year 11 (16-17 years old) students with a Texas Instruments TI-92 CAS calculator in 1998. There were three teachers and classes; A and B at one school, and C at another. Teachers B and C were invited to participate in the study as a consequence of their participation in professional development activities related to technology. Teacher A, who taught the only parallel class to teacher B, volunteered as a personal challenge. We reported (Kendal & Stacey, 1999a & 1999b) how the ways each class used the CAS impacted on their understanding of calculus and was related to their teacher's ″privileging″ (see below).

In 1999, the opportunity arose to repeat the 1998 study with Teachers A and B and their next cohort of Year 11 students. This enabled us to observe more closely the lessons of two teachers whom we knew were distinctly different, using refined assessment instruments. It also enabled observation of the teachers using the CAS calculators for the second time, looking for the features of teaching and of privileging that were stable or changed. This study clearly identifies the teaching context including the teachers and constraints. It reports on the types of differentiation competencies acquired by each class and relates differences between the classes to observed differences in teacher privileging.

Teacher privileging in a technological environment

Privileging is a term adopted by Wertsch (1990) who explained how different forms of mental functioning dominate in different socio-cultural contexts. Privileging is a construct to describe a teacher's individual way of teaching and includes decisions about what is taught and how it is taught. Teaching decisions are influenced by a variety of ″context″ factors such as the teachers' pedagogical content knowledge, teachers' beliefs together with a range of school related cultural factors (Fennema & Franke, 1992).

In a comprehensive literature review of teachers' beliefs and conceptions of mathematics, Thompson (1992) discusses the importance of teachers' conceptions of mathematics to their instructional practices. Several studies are reported where teachers' professed beliefs about the nature of mathematics are consistent with their teaching practices, including her 1984 study that concludes:

> Although the complexity of the relationship between conceptions [of the nature of mathematics] and practice defies simplicity of cause and effect, much of the contrast in teachers' instructional emphases may be explained by differences in their prevailing views of mathematics.(p. 119)

Hoyles (1992) made similar observations in relation to a Microworlds project which explored the ″interactions of the teachers with the computer activities . . . and the ways they incorporated them into their practice [provided] a window on their views and beliefs about mathematics teaching″(p.39). Noss and Hoyles (1996) observe that ″there is a mutually constructive relationship between what teachers believe and what they do″(p.201). Additional factors, related to the teacher's attitude towards technology, also impinge on instructional choices (Thomas, Tyrrell, & Bullock, 1996).

Simmt (1997) observed six teachers and concluded that they used graphics calculators as an extension of their normal teaching practices. They used similar activities, but their differing conceptions of mathematics affected how they followed up those activities with questions and summary notes. Jost (1992), in a comparative case study of teachers' beliefs and practices teaching calculus, found that teachers who believed the purpose of the graphical calculator was computational tended to view learning as listening while teachers who believed the purpose was instructional adopted student-centred teaching approaches. Tharp, Fitsimmons and Brown Ayers (1997) studied 261 teachers participating in a support course for introducing graphics calculators. They noted that teachers generally came to see the graphics calculators as enhancing understanding and promoting exploration. However, teachers with a rule-based conception of mathematics tended to control the amount and type of calculator use by students, and tended to believe that the graphical calculators may hinder learning.

In turn, instructional emphases are reflected in student achievement. Some studies are now appearing which link instructional choices of teachers using technology to student learning outcomes. Keller, Russell, and Thompson (1999) report on a study where use of a CAS calculator and student-centred instruction had a positive effect on students' calculus exam scores. Teaching mathematics with technology opens up new questions about how teachers' beliefs and conceptions, and consequent instructional choices and technological privileging, impact on student learning outcomes.

Analysis of the lessons of Teachers A, B and C in the 1998 study (Kendal & Stacey, 1999a, 1999b) indicates that three aspects of privileging impacted significantly on student learning; teaching approach, emphasis given to different representations of differentiation, and use of technology. Teacher A privileged procedures for standard tasks and symbolic algebra using technology. In class and in assessments, Class A used computer algebra more frequently, was more successful with symbolic items, and less successful with conceptual items. Teacher B privileged conceptual understanding and by-hand algebra while Teacher C privileged conceptual understanding and calculator graphical methods. Both Classes B and C were more successful than Class A with conceptual items. Class B was more successful with by-hand algebra whereas Class C used graphical (non-calculus) methods more frequently and was able to solve many of the test items using conceptually simpler graphical methods as an alternative to symbolic procedures. These three aspects; teaching approach, emphasis given to different representations of differentiation (calculus content), and use of technology, are the key aspects of privileging featured in this paper.

METHODOLOGY

As noted above, Teachers A and B from the 1998 study (Kendal & Stacey, 1999a & 1999b) volunteered to repeat the study in 1999. Both teachers helped revise the twenty-lesson introductory calculus program and greater emphasis was placed on the concept of derivative in numerical, graphical and symbolic representations and links between them. The teachers subsequently taught the program to (new) Classes A and B, shared ideas about lessons, and used the common teaching program, lesson notes and work sheets that they helped prepare, except for two revision lessons at the end of the unit, which they prepared separately. Both teachers were now experienced in teaching with the TI-92 calculator, having used it in the prior study. The first author observed and audiotaped all the lessons, maintained a journal, and interviewed the teachers individually before and after the program. Ten weeks after the completion of the unit, the teachers were interviewed individually by the first author.

Before undertaking the calculus unit, the thirty-three Year 11 female students (aged approximately 17 years) in Classes A and B learned how to use the TI-92 which was theirs to use during the teaching program for all class work, homework, and tests. In contrast to the first study, where the attainment levels of students in the three classes were approximately normally distributed and evenly matched, the two 1999 classes had different distributions of attainment. School assessments over two years showed that both classes had a majority of lower attaining students. However Class A had a larger number of highly competent students, which resulted in average school attainments higher than that of Class B in most topics including the directly relevant topics of algebra and graphs. This was verified by a pre-test.

Students completed questionnaires, challenging assignment questions and completed two written tests. The first author also conducted task-based interviews with fifteen students at the end of the unit, although these are not reported on in this paper.

Testing in 1998 used standard school questions and in consequence over-represented symbolic items, so a major aim of the repeat study was to assess competence in calculus in a balanced way across all three representations (the numerical, graphical and symbolic). A D*ifferentiation* C*ompetency Framework* was developed to provide direction and organise data collection, analysis and interpretation. The framework identifies competence within and between pairs of representations. It is fully described by Kendal & Stacey (submitted). The framework is concerned with two parts of mathematical modelling with differentiation:

- *Formulating* the problem in terms of differentiation either exactly or approximately at a point, or as a derivative function.
- *Interpreting* the derivative either in natural language, in a real world setting (e.g., speed) or in another representation.

Formulating and interpreting can be carried out numerically, graphically or symbolically. Sometimes this is within one representation and sometimes, data from one representation needs to be translated for formulation or interpretation in another. Thus, in total, the Differentiation Competency Framework consists of eighteen basic competencies involving formulation and interpretation processes, six within a single representation (3 per process) and twelve with translation between two representations (6 per process).

A *Differentiation Competency Test* (DCT), consisting of eighteen items, was designed to test each of these basic differentiation competencies. Sample items are given in Figure 1. In marking DCT items procedural errors (such as “careless” algebraic, graphical, calculation, or transcription mistakes) were ignored, as the intention was to assess conceptual understanding. Time available for testing limited the number of items to one per competency, so results about individual competencies need to be interpreted cautiously and in the light of information from other sources. However, stronger data is available from grouping items by process or representation. A *Preference of Representation Test* was also used to assess students’ preferences for representations, their ability to carry out accurate symbolic differentiation, and their problem solving skills. Where appropriate the results of some Preference for Representation Test items are included in this paper but it is not fully reported here.

Formulation | **Interpretation**

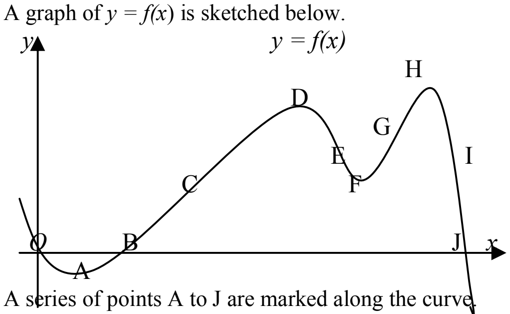
Within a representation

Q.1 [Formulation *within* the symbolic representation]
Find the derivative of $y = x^5 + 4x^3 - x - 10$

Q.2 [Formulation *within* the graphical representation]
Use a graph of
$y = x^2 + x - 10$ to find the gradient of the curve at $x = 3$.

Q. 3 [Interpretation w*ithin* the numerical representation]
At 1.00p.m., the rate of change of the temperature of your house is +3 degrees Celsius (°C) per hour. Immediately after 1.00p.m., is the temperature most likely to: decrease, stay the same, or increase? Give a reason for your answer.

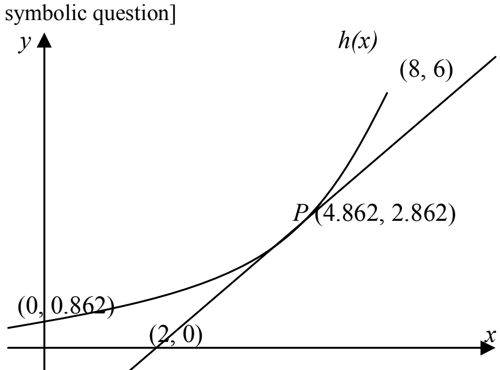
Q. 4 [Interpretation w*ithin* the graphical representation]
A graph of $y = f(x)$ is sketched below.



A series of points A to J are marked along the curve. Consider the statements below and decide if they are true or false.
The gradient of the curve at F is greater than at B.
The gradient of the curve at A is greater than at H.
The gradient of the curve at I is less than at F.
The gradients of the curve at O and J are approximately equal.

Translation between two representations

Q. 5 [Translation of graphical formulation to answer a symbolic question]



The graph of the function $h(x)$ is sketched above. The tangent at the point P on the curve of $h(x)$ has also been drawn. Find the value of the derivative of $h(x)$ at P.

Q.6 [Translation of numerical formulation to answer a symbolic question]
The values of a function close to $x = 5$ are shown in the table below.

x	4.997	4.998	5.000	5.001	5.002	5.003
$f(x)$	15.470	15.482	15.500	15.508	15.515	15.520

Find the best estimate of the derivative $f'(x)$ at $x = 5$.

Q.7 [Translation for interpretation of numerical derivative as symbolic derivative]
An eagle follows a flight path where its height depends on the time since it flew out of its nest. The rule for finding the height of the bird (H in metres) above its nest is a function $f(t)$ of t the flight time (in seconds).
Five seconds after take-off, the 4kg eagle was observed to be 100m above its nest and climbing at the rate of 3 metres/second.
What is the value of $f'(5)$?

Q.8 [Translation for interpretation of symbolic derivative as graphical derivative]
The derivative function of $f(x)$ is given by
$f'(x) = x^3 - 5x + 3$.
What is the gradient of the tangent to the curve $y = f(x)$ when $x = 1$?

Figure 1. Sample questions from DCT, testing formulation and interpretation competencies of differentiation within and between (with translation) representations.

THE TEACHING PROGRAM

Role of the CAS Calculator

The 20 lesson calculus unit relied on CAS facilities of the TI-92, an advanced calculator that made readily available numerical, graphical and representations of derivative, and easy determination of limits and complex symbolic derivatives. With the TI-92, it is also relatively easy to move flexibly between these representations using its split screen facility. The calculators were available at all times including for both tests. Compared with the first study, the strong conceptual focus of the curriculum resulted in an overall reduced emphasis on use of symbolic algebra. This is discussed later in this paper.

During the teaching program, the CAS calculator was used both *functionally* and *pedagogically,* terms used by Etlinger (1974). Pedagogically it was used for concept development in the following ways:

- showing local linearity of curves, zooming in and demonstrating the apparent co-incidence of the zoomed-in curve and the tangent,
- showing the relationship between gradients of secant, curve and tangent by demonstrating, with a dynamic program, that the limit of the secant line is the tangent to the curve at the point,
- plotting gradients/derivatives of points as a function to predict or check the derivative rules,
- building up patterns amongst calculated gradients and/or derivatives to guess symbolic rules,
- using the split screen to emphasize the links between different representations.

Functionally, it was used to carry out routine procedures including: differentiating algebraic expression, obtaining limits for a first principles approach, finding numerical difference quotients, and algebraic manipulation. A menu item was also used to find the gradient of the tangent at a point. The calculator obtains this from a difference quotient, so it is an approximation, but in practice in normal school questions, it is indistinguishable from the exact value.

SAMPLE LESSON 1

To illustrate differences in the approaches of the two teachers, two 45-minute lessons are described in detail. The aim of the first lesson is to teach the general rule for differentiating a power function (i.e. that $d(ax^n)/dx = n.ax^{(n-1)}$) and to apply it to positive and negative powers. In an earlier lesson, the gradients of tangent lines to the curve $f(x) = 3x^2$ had been found at $x = \{-2, -1, 0, 1, 2\}$ using CAS. This had been compared with the results from the one-line CAS symbolic generation of derivatives and substitution (given a set of x values), ***d*****(3*x*^2,*x*) / *x* = {-2, -1, 0, 1, 2}**, thereby linking gradients of functions with symbolic derivatives. The notation *d*(3*x*^2,*x*) indicates the differentiation of 3*x*^2 with respect to *x* and the notation / *x* = {-2, -1, 0, 1, 2}, indicates that the derivative is to be evaluated at each of these points.

Outline of Teacher A's Method

Teacher A presented carefully planned mathematical content with a lecture and demonstration style. Students observed the teacher carry out the CAS procedures, which they imitated with help from a flow chart (see Figure 2) that the teacher had written on the white board.

- Using the TI-92 displayed on an overhead projector, Teacher A generated a set of derivative values for $f(x) = 3x^2$ symbolically, and listed the values of the derivative in a table on the board (See Figure 2).
- Students were invited to guess the rule and a group response of "Multiply by 6" for the derivative pattern ($6x$) was accepted.

- The demonstration was repeated for $f(x) = x^3$ and a student guessed the pattern $3x^2$.
- Without generating further tables, and with minimal student input, the teacher quickly wrote on the board the derivative rules for $f(x) = x^4, 3x^5, 3x^3$.
- A summary table of $f(x)$ and $f'(x)$ was constructed and one student identified the derivative rule for the polynomial power as $n.ax^{n-1}$.
- An example requiring finding $f'(2)$ for $f(x) = 3x^3$ was worked with and without CAS. Firstly, the formula $f'(x) = n.ax^{n-1}$ was applied and $x = 2$ substituted. Then the differentiation and substitution in a one-step CAS routine was performed, which was also recorded on the board using a flow chart (see Figure 2). Most students imitated the CAS procedure then copied down the detailed board notes.
- Practice problems were set including the differentiation of $1/x^5$, the first negative power encountered, and the students began to work on them using the formula or CAS. Unfinished problems were set for homework.
- When a student asked for help, the teacher (as usual) worked another example.

$f(x) = 3x^2$		$f(x) = x^3$			
x	f ′(x)	x	f ′(x)	f(x)	f ′(x)
-2	-12	-2	12	$3x^2$	$6x$
-1	-6	-1	3	x^3	$3x^2$
0	0	0	0	x^4	$4x^3$
1	6	1	3	$3x^5$	$15x^4$
2	12	2	12	$3x^3$	$9x^2$
Multiply by 6		Pattern $3x^2$		ax^n	$n.ax^{n-1}$

Flow chart for TI-92 procedure
Push buttons in given order

1. F3
2. 1 & Enter (E)
3. (3 x ^ 2, x) (2nd K) /x = {-2, -1, 0 ,1, 2} & E

Figure 2. Teacher A’s board notes.

Outline of Teacher B’s Method

Teacher B explored carefully planned mathematical content through a teacher-led class discussion, drawing on individual student’s contributions.

- Students generated the values of the derivative of $f(x) = x^2$ at five points using the CAS command to differentiate and then substitute the given set of x values in one line, ***d*(*x*^2,*x*) / *x* = {-2, -1, 0, 1, 2}** The teacher moved around the classroom checking individual's screens and giving help with the calculator.

- Student generated values were written on the board (see Figure 3) and individual students were challenged to find and explain how the pattern was developed, and to verify the resultant symbolic pattern, derivative = $2x$.
- The procedure was repeated for $f(x) = 2x^2$ and $3x^2$. For $f(x) = ax^2$ the derivative rule $f'(x) = 2ax$ was spontaneously suggested by one student and confirmed by others.
- A student observed that quadratic function gave a linear derivative. This was observed from a sketch of $f(x) = 2x^2$ drawn on the board, and awareness that the derivative, $4x$, represented a linear function and straight line.
- Another student posed the question ″What about x^3?″ Students generated a table of derivative values using the TI-92 and guessed the derivative rule for $f(x) = x^3$.
- Teacher B asked individuals to predict the derivative rules for $f(x) = 2x^3$, $f(x) = 3x^3$, and eventually for $f(x) = ax^3$.
- A student observed that a cubic function gave a quadratic derivative and the teacher helped all the students understand why. (Diagrams were drawn on the board and links with gradients made). The algebraic patterns were revised again.
- Students generated the derivative rule for $f(x) = x^4$ assisted by a tables of derivative values which they generated using the TI-92 (see board notes in Figure 3).
- Individual students predicted the derivative rule for $f(x) = 2x^4$, $f(x) = 3x^4$, and eventually for $f(x) = ax^4$.
- Using the summary table of $f(x)$ and $f'(x)$ for the general expressions, for quadratic, cubic and quartic polynomials, after class discussion a student volunteered the derivative rule for the polynomial ax^n.

- Teacher B carefully explained the $f'(x)$ notation and checked students' understanding by nominating individuals to explain to the class how to use it. He again revised the general rule for differentiation of a polynomial function.
- Students progressively copied the notes and sketches from the board including a section devoted to alternative notations, $f'(x)$ and dy/dx.
- The special functions $f(x) = ax$ and $f(x) = c$ were sketched on the board, the gradients determined, and the outcomes linked to the formula $n.ax^{n-1}$.
- Examples using the rule were worked by-hand, including $f(x) = 5x^7$ and $f(x) = 1/x^4$.
- Practice problems were set for homework and included the differentiation of $1/x^5$.
- Students were instructed to do the items by-hand and to check their answers by repeating the items with the TI-92.

$f(x) = x^2$		$f(x) = x^3$		$f(x) = x^4$		$f(x) = ax^n$		Patterns	
x	$f'(x)$	x	$f'(x)$	x	$f'(x)$	x	$f'(x)$	$f(x)$	$f'(x)$
-2	-4	-2	12	-2	-32	$a x^2$	$2.ax$	quadratic	linear
-1	-2	-1	3	-1	-4	ax^3	$3.ax^2$	cubic	quadratic
0	0	0	0	0	0	ax^4	$4.ax^3$	quartic	cubic
1	2	1	3	1	4	ax^n	$n.ax^{n-1}$	Examples	
2	4	2	12	2	32			1. $f(x) = 5x^7$	
$f'(x) = 2x$		$f'(x) = 3x^2$		$f'(x) = 4x^3$		$f'(ax^n) = n.ax^{n-1}$		$f'(x) = 35x^6$	
$f(x) = 2x^2$		$f(x) = 2x^3$		$f(x) = 2x^4$		Special cases		also for	
$f'(x) = 4x$		$f'(x) = 6x^2$		$f'(x) = 8x^3$		with diagrams		$y = 5x^7$	
$f(x) = 3x^2$		$f(x) = 3x^3$		$f(x) = 3x^4$		$f(x) = ax = ax^1$		$dy/dx = 35x^6$	
$f'(x) = 6x$		$f'(x) = 9x^2$		$f'(x) = 12x^3$		$f'(x) = a$		2. $f(x) = 1/x^4$	
$f(x) = ax^2$		$f(x) = ax^3$		$f(x) = ax^4$		$f(x) = a = ax^0$		$f(x) = x^{-4}$	
$f'(x) = 2.ax$		$f'(x) = 3.ax^2$		$f'(x) = 4.ax^3$		$f'(x) = 0$		$f'(x) = - 4x^{-5}$	

Figure 3. Teacher B's board notes (diagrams, notation section, with some examples omitted).

SAMPLE LESSON 2

This lesson introduced maximum and minimum problems using the following standard textbook problem:

From the corners of a rectangular piece of cardboard, 32 cm by 12 cm, square sides of side, x cm, are cut out and the edges turned up to form a box. Find the value of x if the volume of the box is a maximum.

Outline of Teacher A's Method

- Firstly, Teacher A drew a diagram of a rectangle on the blackboard (representing the cardboard) and with one student's help wrote down the rule for the volume of the box, $V = x(32 - 2x)(12 - 2x)$.
- Next, the differentiation and how to solve for zero derivative was demonstrated, in a two step, one line CAS procedure [solve($d(x(32 - 2x)(12 - 2x),x)=0,x$)].
- Finally, Teacher A demonstrated how to set up a table of slope values to decide the nature of the stationary points, using CAS to calculate the derivative and substitute nearby values between the zeros into it. A rough graph was constructed from the slope values with focus on the zero gradients.
- The appropriate x value was noted, and the second value discarded.
- The students copied the notes from the board.

Outline of Teacher B's Method

- Teacher B constructed a cardboard box and discussed the problem with the students using a diagram on the blackboard.
- Next, Teacher B led a class discussion which generated the dimensions of the box (in terms of x) and its volume: $V = x(32 - 2x)(12 - 2x)$.
- The students assisted the teacher in differentiating and finding the zeros of the derivative by-hand. This resulted in location two values of x where the derivative is zero.

- Students graphed the function on their calculators, and Teacher B illustrated the algebraic result graphically from a white board sketch of the graph.
- Finally, the necessity to ignore one of the algebraic solutions (for the real box) was discussed.

TEACHER PRIVILEGING

These two Sample Lessons clearly highlight differences between the teachers with respect to their teaching approaches, emphasis given to different representations, and use of technology, the three aspects of privileging of interest in this paper. Both teachers worked through the planned teaching program that they had helped develop, but during its implementation they made pedagogical choices about what was important to teach and how to teach it, what was important to emphasize including representations of differentiation, and how to incorporate CAS into their lessons. These choices about working with CAS were consistent with their fundamental conceptions of mathematics, and innate privileging.

Differences in teaching approaches

From a range of possible classifications of teachers' conceptions of mathematics and teaching approaches, Thompson (1992) identifies Kuhn and Ball's (1986) model of mathematics teaching ″as constituting a consensual knowledge base regarding models of teaching″(p.137). Their model identifies four different teaching approaches:

> *Learner focused*: mathematics teaching that focuses on the learner's personal construction of mathematics knowledge;
>
> *Content-focused with an emphasis on conceptual understanding*: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;
>
> *Content-focused with an emphasis on performance*: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures; and

> *Classroom focused*: mathematics teaching based on knowledge about effective classrooms. (Thompson, 1992, p.136)

Based on this model, Teacher A's teaching approach is best characterized as *content-focused with an emphasis on performance*. He emphasized mathematical rules and procedures, generally lectured the students and demonstrated procedures, had relatively little interaction with students, and conducted little interactive class discussion. He ″automatized″ computational procedures and taught Class A to respond to a range of specific data cues, the words and/or context clues of the question. In the interview he explained:

> I'd say, when you see these words it means between two points, and when you see this word that means at a point, and a maximum means that you let the derivative equal zero . . . [I am] giving them strategies.

Teacher B's teaching approach is best characterized as *content-focused with an emphasis on conceptual understanding* described by Thompson (1992, p.136) with its distinguishing feature, the ″dual influence of content and learner. On one hand, content is focal, but on the other, understanding is viewed as constructed by the individual″(Kuhn & Ball, 1986, p.15). Teacher B perceived mathematics as a logical coherent body of knowledge waiting to be discovered by students. He taught for conceptual understanding and encouraged students to develop their intuitive ideas about rate of change, slope and the limit concept. His style of teaching was student-centred. He orchestrated interactive class discussions (in which each student participated and shared ideas) to explore the mathematical content and logical construction of the concept being developed. Each student appeared to conjecture, analyze, negotiate meaning with other students in the class, make decisions, draw conclusions, and prove ideas. Teacher B made regular checks that all the students understood what was being discussed (e.g., in Sample Lesson 1 above, how the derivative rules fitted the patterns of numbers).

In summary, Teacher A (*content-focused with an emphasis on performance)* privileged knowledge of routine procedures and rules (using a lecture style of teaching) while Teacher B (*content-focused with an emphasis on conceptual understanding)* privileged conceptual understanding of mathematical ideas and student construction of meaning (using student-centred style of teaching). These two different teaching approaches are consistent with Joss' (1992) observations that teachers who used calculators for procedures viewed learning as listening and those who used calculators for learning used student-centred teaching styles.

Privileging of representations

Teacher A supported use of all three representations but had a strong personal preference for the symbolic representation. This is illustrated by Sample Lessons 1 and 2 above, which are both entirely in the symbolic representation. The obvious link between the graphical and symbolic, evident in Teacher B's lessons, is not evident for Teacher A. In an interview, he indicated that although he had a strong preference for the symbolic representation he began to use the graphical (and numerical) representations of derivative when he realized that the CAS calculator would give an "exact" gradient for the tangent to a curve. Although the gradient is actually an excellent approximation, Teacher A thought it was exact.

Teacher B privileged both symbolic and graphical representations of derivative. In an interview, Teacher B stated that he strongly directed his students towards the symbolic representation because of his personal beliefs that it was the most important and useful representation of derivative, and that by-hand algebra was crucial for understanding. He also revealed his conviction that his current cohort of students, being less able than other groups he had taught, would not cope successfully with more than one representation. Hence his primary teaching focus was the symbolic representation. Thus, the relatively good symbolic competencies of Class B (see below) are particularly significant in light of their weaker algebraic background and lower school performances. However, he frequently used a graphical representation to interpret the symbolic derivative visually. Both Sample Lessons 1 and 2 illustrate this. Teacher B frequently used physical representations of slope and speed, and linked differentiation ideas to the real world. He consistently and deliberately gave the symbolic derivative meaning by linking it to the slopes of tangent through enactive arm movements and to rates of change (thereby linking the symbolic representation to the graphical and numerical representations).

Further evidence for different privileging of representations came from two other observations. First, each teacher allocated different amounts of teaching time to connecting different representations. Teacher A allocated a total of 86 minutes to the graphical-numerical connection and Teacher B allocated 5 minutes. For the graphical-symbolic connection, Teacher A allocated 60 minutes and Teacher B allocated 185 minutes. Second, the teachers organized their own revision programs differently. Teacher A spent one lesson revising CAS procedures across the three representations and a second, by-hand procedures. Teacher B used 1995 – 1997 revision sheets, previously prepared for use with the former traditional curriculum, and mainly involving by-hand symbolic procedures.

Privileging related to use of technology

In an interview, Teacher A indicated his dissatisfaction with the normally used non-CAS graphics calculator, so it was a surprise that during the first trial he stated a strong preference for the symbolic capability on the calculators and he permitted his students free use of the symbolic calculator. In the second trial, he continued to privilege the use symbolic differentiation with technology and extended his functional use of CAS to differentiate graphically and numerically. He embraced the new procedures and taught students to use them. Teacher A may have been expected to resist relinquishing control of valued ″by-hand″ routine procedures, but rather he seemed comfortable including a whole new range of technical procedures in his teaching practices. He demonstrated use of the technology to the students using an overhead projector, listed step by step routines for each calculator procedure, and then allowed students free use of the calculator. Sample Lesson 1 demonstrates this. At the interview, Teacher A commented:

> I used it . . . because it was so easy. I hooked it up at the beginning of the lesson and used it much more than I would use a graphics calculator in the classroom. We just used it (the TI-92) all the time, . . . routine procedures like the product rule and the chain rule, umm, yes, . . . They actually hadn't made the distinction that there were functions that they could only find the derivative of on their calculator.

His preference for the symbolic representation had an unexpected consequence in giving a strong emphasis to aspects of the numerical representation of differentiation. He showed his students how to use CAS substitution procedures to find ordered pairs, for use in a difference quotient calculation, for an "excellent" approximation to the gradient of the curve and gradient of the tangent. He therefore privileged the link between the numerical representation and the graphical representations of derivative. He also used CAS pedagogically to link numerical and graphical representations of derivative.

In contrast, Teacher B's focus was on conceptual understanding and he privileged pedagogical use of CAS for activities that linked the symbolic and graphical derivatives. At the interview, he said a highlight was

> getting the tangent idea through to them, what the gradient actually represents, what the derivative represents and the relationship between them - I think we've done very nicely with the calculator.

He allowed Class B to use the graphics capabilities of the calculator, but restricted use of symbolic algebra to procedures that supported conceptual understanding. For example, in Sample Lesson 1, Class B students used CAS to quickly and accurately gather symbolic data from which the rules of differentiation were inferred. However, they did the practice exercises by hand. Teacher B explained his position thus:

> I liked the routine procedures. For example, when you're trying to induce symbolic rules, you haven't got all that time wasting. You can very nicely do a lot of the algebra so simply on the calculator and you're avoiding wasting time doing a lot of repetitive calculations.
>
> It's [the CAS] good for discovery because it takes a lot of the hack work out of teaching for understanding but you still need to teach pen and paper skill. I think there's certain skills that the kids have to have, even if you can use the technology to do it. I think the kids have to have the (algebraic) skills as well, without the technology. I think that's essential for their understanding. It's not sufficient to just use the calculator; they have to have the understanding, what's behind it.

Teacher B believed that symbolic routines carried out in a ″black box″ did not assist understanding and insisted that (apart from the use described above) students perform algebraic procedures by-hand. He also warned his students not to become dependent on

CAS since they would not be permitted to use it for exams after this experimental unit. In contrast, he strongly supported use of graphics calculators (permitted in the state examination system) and had downloaded a set of programs onto his students' graphics calculators including some symbolic programs (e.g., solution of quadratic equations).

Changes in curriculum and teaching with experience

During the repeat study, less time was available for the teaching program (due to school time constraints), so fewer lessons were devoted to concept development prior to learning symbolic differentiation procedures. In addition, the teachers gave less emphasis to the construction of tables for numerical differentiation since the students in the previous study had experienced a variety of time-consuming technical difficulties.

During their second teaching experience the teachers made changes to technology privileging (discussed above). Table I summarizes these changes and the reasons for them. These reasons have been constructed by the authors from the evidence in the interviews. While neither teacher changed his teaching approach (method or style), both made changes to their privileging of technology in ways that were consistent with their fundamental conceptions. Teacher A increased his use of technology to include new procedures for numerical and graphical differentiation and pedagogical activities that linked the numerical and graphical representations. In contrast, Teacher B reduced student use of CAS for symbolic and numerical differentiation.

Table I
Changes in Technology Privileging (and Reason for Change) by Each Teacher During Trial 2

Changes in technology privileging	Reason
Teacher A	
• Adopted numerical and graphical differentiation procedures	Realized numerical and graphical procedures would be tested and came to believe that numerical and graphical differentiation procedures were "exact".
• Incorporated pedagogical activities that linked the numerical and graphical representations	Liked demonstrating ideas with technology
• Gave more emphasis to by-hand symbolic differentiation	Responded to influence of Teacher B
Teacher B	
• Reduced use of CAS for symbolic differentiation and rejected use of CAS for numerical differentiation	Believed that his less able cohort of students would not cope with three representations

LEARNING OUTCOMES AND DIFFERENCES BETWEEN CLASSES

In this section, differences on the achievement of students in Classes A and B is reported with respect to success on the 18 DCT competencies and on various groups of competencies (viz., each representation and each type of competency by representation). Differences in class achievement and frequency of CAS use on the six symbolic items on the Preference for Representation Test is also reported. Then in the final section of the paper, the information about the teaching and the teachers is considered together with the information on learning so that the consequences of the teachers' privileging can be examined and discussed.

Differences in attainment of the competencies

The classes achieved a similar average number of competencies on the DCT: the average for Class A (N = 14) students was 8.3 (SD 3.6) and for Class B (N = 19) it was 8.5 (SD 3.3). This almost identical class achievement of the 18 competencies was unexpected since Class A had a higher proportion of highly academically competent students and would have been expected to do better. However, the overall low achievement of each class was consistent with previous school testing.

Differences in superiority on different types of competencies

Average class achievement for different types of competencies (the percentage of students in each class who demonstrated the competencies) was determined. On the group of six competencies associated with each representation, both classes achieved best on the symbolic representation particularly Class B whose achievement was superior to Class A. Superiority is reported if there is a greater class achievement of at least 10%. Both classes achieved second best on the group of graphical representation competencies and least well on numerical competencies.

Table II below, displays the number of competencies on which each class was superior in each type of competency by representation. For example, Class A was superior (by 20.3%) on both of the formulation with translation competencies in the numerical representation. None of the differences in class achievement was statistically significant, but the pattern of differences reflected the different teaching emphasis that we observed, increasing our confidence in the results.

Table II shows that:

- Overall, Class B was superior on 5 competencies and Class B was superior on 4.
- Class A was superior on both N and G competencies, Class B on S competencies.
- Overall, Class A's superiority was on formulation with translation competencies, whereas Class B's superiority was on interpretation.

Class A's superiority pattern reflects Teacher A's willingness to embrace the numerical and graphical representations during the second teaching trial and his emphasis on functional routine procedures associated with formulation. Class B's pattern of superiority on interpretation reflects Teacher B's efforts to give derivatives meaning, evidenced in Sample Lessons 1 and 2 and matches well with the emphasis that he gave to conceptual understanding and development of intuitive ideas. The precise patterns of achievement and differences of achievement on individual competencies within these groups are complicated.

Table II
Number of Competencies on which Class Achievement was Superior by Type of Competency, Representation and Class

Type of differentiation competency	Representation: Numerical (N) A	Numerical (N) B	Graphical (G) A	Graphical (G) B	Symbolic (S) A	Symbolic (S) B
• Formulation (3 competencies in total)						
• Formulation with translation (6 competencies in total)	2		1			1
• Interpretation (3 competencies in total)		1		1		
• Interpretation with translation (6 competencies in total)			1			2

Differences in frequency of CAS use

Although the two teachers planned to use CAS similarly, Teacher A allowed students free use of CAS, whilst Teacher B restricted it. Observations of lessons established that Class A students used their calculators more frequently than Class B students. This was consistent with the teachers' comments reported above. Six items from the Preference for Representation Test required differentiation of functions such as $g(x) = x^3\ (2x + 1)^2$, $h(x) = (4x^2 + 9)^7$ and $f(x) = 1/x^6$, which would be done by hand using the chain and product rules.

36% of Class A used CAS to differentiate compared with 9% of Class B. Clearly the use of computer algebra advantaged Class A since its overall success rate on these items was 71% compared with Class B's 57%. Pre-testing had shown Class B had weaker algebraic skills initially and as a class they did not use CAS to compensate for this.

DISCUSSION

This case study approach provides an example of how two practising teachers incorporated technology into their pedagogy in different ways, consistent with their different beliefs. Teaching introductory calculus with a CAS calculator provided the teachers with a viable range of ″in practice″ pedagogy previously accessible only ″in principle″ in the classroom. Both teachers intended to teach the same curriculum material in the same way but they made different pedagogical choices from the range of options. They had fundamentally different conceptions of mathematics with associated teaching approaches and innate privileging which influenced their particular choices while using technology, about what to emphasize, and how to incorporate the graphical and symbolic algebra capabilities of the calculator into their lessons. These choices directly affected what they taught, how they taught it and what their students learned. Although both classes achieved almost identical achievement of differentiation competencies on the DCT, they showed quite different strengths.

Teacher A (*content-focused with an emphasis on performance)* had expressed dissatisfaction about teaching with graphics calculators, and used them only when essential in his standard teaching (probably because of a preference for "exactness"). This was in accordance with the behaviour of the rule-based teachers in the study reported by Tharp et al. (1997). However, with CAS available, Teacher A enthusiastically embraced use of the technology. It appears that he was able to increase his repertoire of ways of teaching procedures for achievement in a manner that was consistent with his innate privileging of teaching rules and procedures and preference for exactness. He added a set of functional CAS procedures for "exact" differentiation in the numerical and graphical representations to the rules he already used for exact symbolic differentiation. In consequence, Class A was superior at using CAS for symbolic differentiation and with translating between representations for formulation.

In contrast, Teacher B (*content-focused with an emphasis on conceptual understanding*) used the symbolic capability of the CAS less in the second study than in the first and turned back towards traditional practice. He privileged conceptual understanding of mathematical ideas and student construction of meaning (both with and without technology) and Class B students were superior on more interpretation competencies. Teacher B used the CAS calculator principally for pedagogical purposes. Possibly the lack of functional use was because this introductory course did not provide him with sufficient opportunity to utilize the computational capabilities of the CAS calculator as creatively as he did with the graphical calculator in his normal teaching practices. Moreover, there was no move within this introductory unit (with its focus on understanding) to extend the computational demands beyond normal by-hand expectations. He gave the symbolic representation the highest status and in consequence, Class B displayed a preference for the symbolic representation and was superior on symbolic competencies, significant when considering their weaker mathematical and algebraic background.

Teacher B's reluctance to give up by-hand algebra leads us to question how important by-hand practice is for success on the essentially conceptual items of the DCT. Although CAS use was optional, symbolic manipulation was hardly needed on the DCT as the algebraic demands of items were simple. Lagrange (1999), working in the French tradition, suggests that ″techniques″ play an important role in conceptual understanding. Techniques in this sense involve both the conceptualization of the steps required and their execution. Is Class B's greater facility with the symbolic representation of derivative due in part to Teacher B's insistence on by-hand practice? In future CAS studies, the French notion of ″techniques″ could be further explored.

In this study neither class developed a truly holistic approach to the concept of derivative. Is there a teaching style, privileging and consequent pedagogical choices under which all students could acquire the complete set of numerical, graphical and symbolic competencies associated with the concept of derivative, within the time allotted in the curriculum? Alternatively, if choices are to be made, which skills and representations and links between them are most important? Technology is providing a growth in options for doing and teaching mathematics. This study has contributed to understanding how greater options will manifest themselves in practice and what the consequences are for students.

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