The Future of the Teaching and Learning of Algebra

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Overview

- Introductory remarks
- Factors for future directions of algebra

- Responding to the rise of numerical mathematics
- Bringing back algebra - can CAS do it?
- Algebra and reasoning and proof
- Discussion points

Influences on future directions….

Current achievement around the world

Social trends

- mass secondary (tertiary) education
- concern for equity
- concern for relevance

New technology

Extensive research on learning algebra

A surfeit of good curriculum ideas ?

QCA : Five year review of standards (GCSE, 1995 - 1999)

"The introduction of multi-step questions and the increased emphasis on manipulative algebra should establish a stronger mathematical background for further study of mathematics. However, the evidence from the scripts suggested that candidates at grade C from either tier displayed none of the characteristics desirable for such further study."

TIMSS data 1996 (13 yr olds)

Find x: $10x - 15 = 5x + 20$

		ACT	42.6 %
Australia	35 %	NSW	39.1 %
Singapore	80 %	VIC	26.6 %
Internat'l	45 %	QLD	35.8 %
		SA	40.1 %
		WA	37.1 %
		TAS	16.9 %
		NT	30.4 %

Number of linear equations with unknown on both sides

Victoria (~ 1997)	Naked equations	Problem situations
Textbook 10A	34	1
Textbook 10B	0	0
Textbook 10C	13	9
Textbook 10D	7	1
Textbook 10E	10	0
Textbook 10F	16	7
Textbook 10G	18	0

Influences on future directions….

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Algebra

- should be a gateway
- very often a wall
- and what is it a gateway to?

An assessment of possibilities for big change for most students ......

Research on learning algebra

Fabulous curriculum ideas

low - transposition didactique?

More curriculum time or emphasis

medium but maybe “head in sand”

New technology

no simple fix; opens avenues for algebra without algebraic thinking

Arithmetic thinking

- Knowns to unknowns
- Unknowns transient
- Equation as formula
- Chains of successive calculations

Algebraic thinking

- Working with unknowns
- Unknown fixed
- Equation as description
- Logical chains of equalities

Algebra and computer technology

- Wonderful new tools e.g. spreadsheets, function graphers, dynamic geometry measurement, ….
- Technology can be used to support traditional methods and existing curriculum
- Technology also challenges the current curriculum
  - to give people usable workplace skills
  - to re-assess value of content taught

Professor Stacey is running a conference and the registration fee is $450. How much GST (10%) does she have to pay?

- Solution 1: $450 is 110% of the pre-GST fee. So GST is $40.91 (450 div by11)
- Solution 2: 450 = x + 10% of x = 1.1x etc
- Solution 3: Spreadsheet guess-check-*improve*

A spreadsheet gives a practical solution

PRICE	GST	TOTAL
100	10	110
400	40	440
410	41	451
409	40.9	449.9
409.1	40.9	450

- Two simple "action" formulas

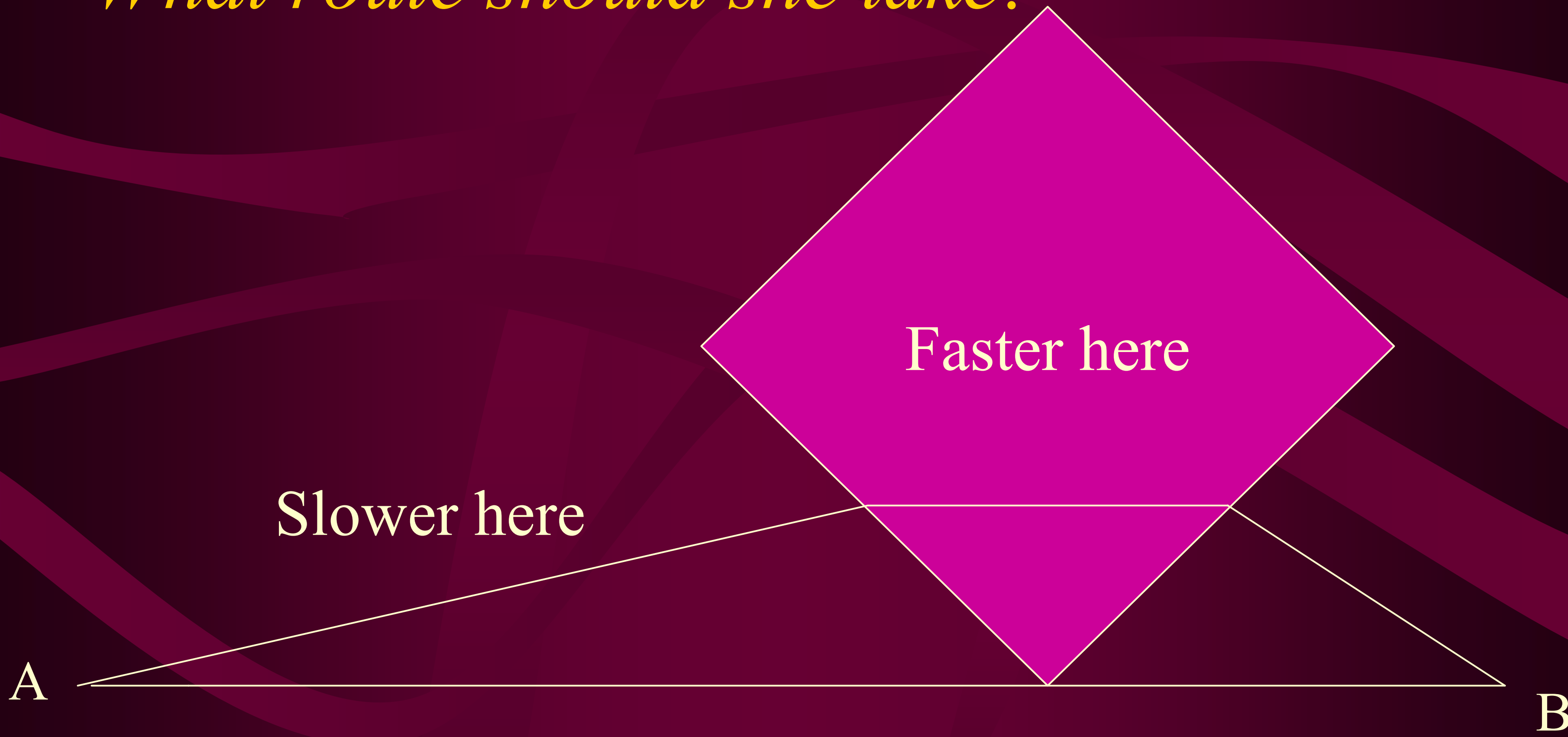
  GST = price / 10

  total = price + *GST*
- Two simple "fill down" formulas
- Very quick guess-check-improve

Orienteering with Kim.

Kim wants to run from A to B through varying terrain.

What route should she take?



Orienteering with Kim.

Kim wants to run from A to B through varying terrain. What route should she take?

- Solution 1: Algebraic formulation, with calculus for optimisation
- Solution 2: Algebraic formulation, with graphical optimisation
- Solution 3: Dynamic geometry formulation, with “dragging” optimisation

Orienteering with Kim.

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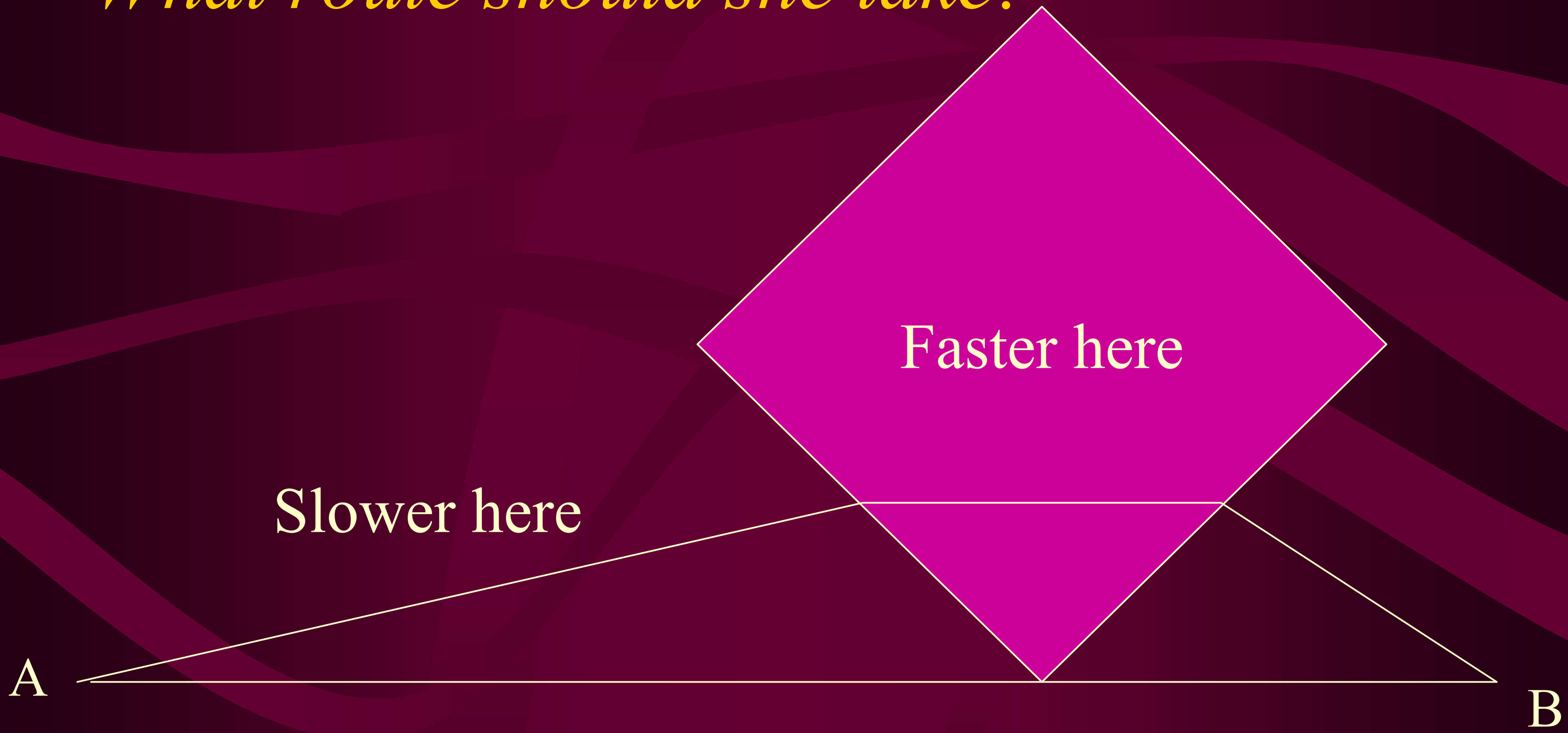
What route should she take?



Is numerical algebra a possibility?

- Numerical (& graphical) methods conceptually easier than algebraic
- Possible features:
  - built around spreadsheet
  - a new “dialect” of maths, with newly acceptable syntax, new primitives (e.g. fill down),..
  - uses methods that have always existed “in principle” but now exist “in practice”
  - guess-check-improve on firm footing
  - need to polish skills (e.g. good use of tables, setting up equation equivalents, etc)

Discussion Points

Is a numerical algebra worth exploring for some/all students?

Challenge: If we want to promote (symbolic) algebraic methods, then we need problems that require them

Bringing back algebra - can CAS do it?

CAS capabilities:

- arithmetic
- tables (spreadsheet)
- symbolic algebra
- function graphing
- calculus
- statistics
- programmable

We don't know what is inside the box now

Possibilities for CAS use:

- *Pedagogical - a teaching tool to strengthen understanding, scaffold skill development etc*
- Enable students with poor manipulative skills to move onto current work
- Re-allocate time saved on teaching routine procedures (e.g. differentiation, *some* eqn solving)
- Allow more realistic/substantial problems (e.g. Kim)

CAS, algebra and reasoning

Find max/min values of $y = x^2e^{ax}$

SOLVEX(d/dx(x^2e^{ax}) = 0)

(x = 0) or (x = -2/a)

Are these max or min? Argue from graph or positive values or … Are there any others?

- Which parts of that reasoning matter most?
- Possible change in examination instructions:

  SHOW ALL WORKING to become EXPLAIN ALL REASONING

R.I.P.A.

Unsatisfactory CAS reasoning?

$(\sin x + \cos x)^2 = 1 + \sin 2x$

$(\sin x + \cos x)^2 =$

$(\sin x)^2 + 2 \sin x \cos x + (\cos x)^2 =$

$1 + 2 \sin x \cos x =$

$1 + \sin(2x)$

Need to distinguish pragmatic from epistemic

There are some chickens and pigs on a farm. Altogether there are 120 legs and 50 heads.

# pigs	# chickens	# heads	# legs
45	5	50	190
40	10	50	180
35	15	50	170
		50	
		50	
5 fewer pigs gives 10 fewer legs ……..if 5 pigs turn into chickens, there are 10 fewer legs ………solution			

There are some chickens and pigs on a farm. Altogether there are 120 legs and 50 heads.

$p + c$	$= 50$	eqn 1	
$4p + 2c$	$= 120$	eqn 2	
$2p + 2c$	$= 100$	eqn 3	(2 x eqn 1)
$2p$	$= 20$	eqn 4	(eqn 2 - eqn 3)
p	$= 10$	eqn 5	(eqn 4 div by 2)

$p + c$	$= 50$	number of heads is 50
$4p + 2c$	$= 120$	number of legs is 120
$2p + 2c$	$= 100$	*if all had 2 heads, there would be 100 heads*
		OR
		if all had 2 legs, there would be 100 legs
$2p$	$= 20$	the extra legs amount to 20 (extras because pigs have 2 extra legs)
p	$= 10$	there are 10 pigs

$$p + c = 50$$

$$4p + 2c = 120$$

$$2p + 2c = 100$$

$$2p = 20$$

p = 10

number of heads is 50

number of legs is 120

if all had 2 heads, there would be 100 heads

OR

if all had 2 legs, there would be 100 legs the extra legs amount to 20 (extras because pigs have 2 extra legs) there are 10 pigs

Algebra in reasoning and proof

- Recapturing the meaning from the algebraic manipulation can be intellectually demanding.
- Recapturing the meaning from the algebraic manipulation is not really helpful.
- Using algebra feels like “calculating”, rather than explaining or reasoning.

Discussion points

- What would a numerical school maths curriculum be like?
  - Content
  - Role of algebra
- Distinguishing the “working” from the “reasoning” and a pragmatic from epistemic purpose
- What intended curriculum gives the best achieved curriculum - how to use all these fabulous ideas?

Thank you

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THOAN - even in a context, order of operations is often not appreciated

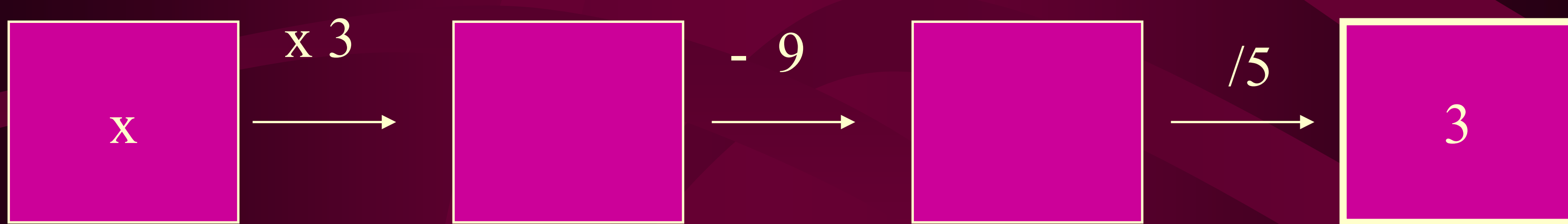
I think of a number, multiply it by 3, take away 9 and then divide by 5. The answer is 3. What was the number I thought of?

$$((x \text{ x}3 -9) / 5) = 3$$

Answer: 8

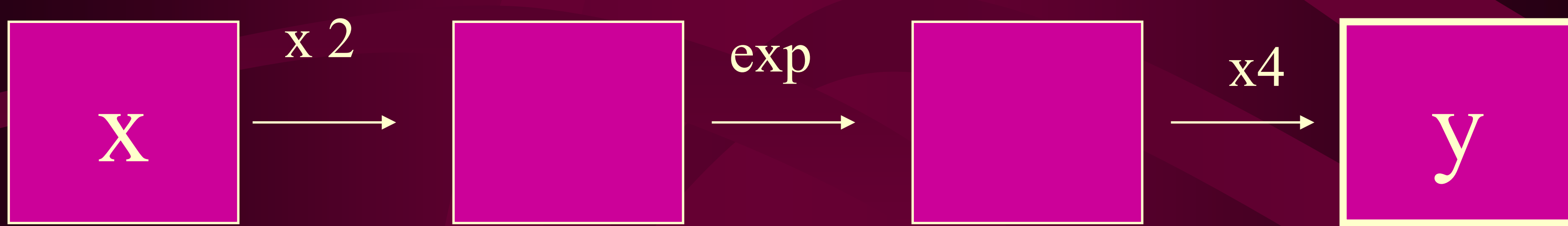
Example of "backtracking"

I think of a number, multiply it by 3, take away 9 and then divide by 5. The answer is 3. What was the number I thought of?



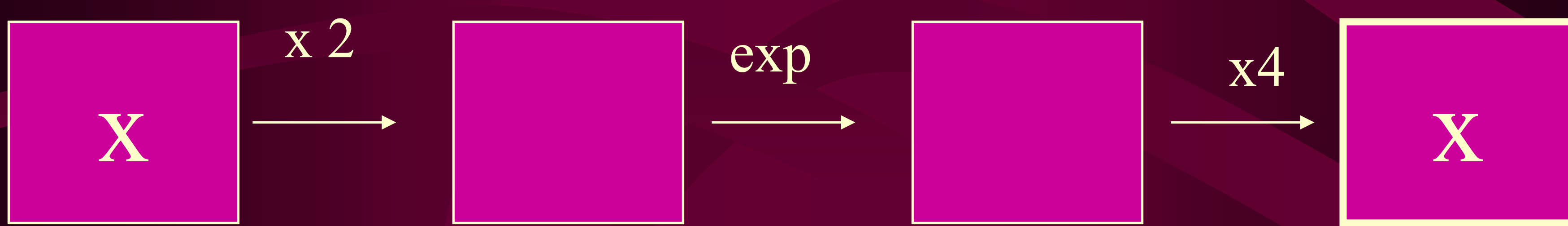
Backtracking (undoing operations) is essential

$$y = 4\,e^{2x}$$



Backtracking (undoing operations) is not enough

$$x = 4\,e^{2x}$$



The rise of "backtracking"

- "We have discovered a way whereby all kids can get all the algebra questions correct!"

"After this lesson the kids laughed at the trivial problems in the text."

(Year 7 teachers, 1981/1988, quoted in MCTP etc)

- 1990s: backtracking spreads up school and becomes VERY complicated (double rows of diagrams etc). By 1997, exclusively taught in some text series even at age 17.
- 2000: Influence of new curriculum document restores algebraic methods

I think of a number, multiply it by 3, take away 9 and then divide by 5. The answer is 3. What was the number I thought of?

x = ((3x5)+9) /3 = 8

(3x(5+9))/3 = 14

(3x5)+(9/3) = 18

(3+(9/5))/3 = 1.6

(3+9x5)/3 = 16

Answers:

8 (correct), 14, 18, 1.6 and more