ACQUIRING THE CONCEPT OF DERIVATIVE: TEACHING AND LEARNING WITH MULTIPLE REPRESENTATIONS AND CAS

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Abstract

For the second time, two teachers taught an introductory program of differential calculus using an advanced calculator. An innovative derivative competency framework was devised, and proficiency in each representation determined using a set of twenty-one different differentiation competencies. Although each class achieved similar overall facility on the test items they exhibited different strengths. More students in Class A became proficient in using the graphical representation and in particular they were better at translating to a graphical representation. Class B developed more proficiency with the symbolic representation and again this was particularly evident in translating to the symbolic representation. These different results reflect different ″privileging″ of the teachers.

Introduction and conceptual framework

Recently, the advent of advanced calculators with graphical, numerical and symbolic computer algebra systems (CAS), has made calculus more accessible to a wider range of students, and provided impetus for research into how students acquire conceptual understanding of differentiation. Tall (1996) describes a set of component differentiation processes/concepts (in different representations) for derivative and Lagrange (1999a) discusses a set of ″schemes of use″ for functions and pre-calculus using CAS.

Kendal and Stacey (1999) report a 1998 teaching trial of introductory differential calculus designed to explore CAS use and assessment. The test items were mainly symbolic and each class made different use of CAS which related to different teacher ″privileging″. A follow-up teaching trial was conducted in 1999 with a new primary objective; to identify specific competencies acquired by students while learning differentiation for the first time using multiple representations of derivative with CAS available. This paper reports the conceptual understanding of differentiation acquired by each class using a set of test items specially designed to measure competence in numerical, graphical and symbolic representations.

A *derivative competency framework* was developed and consists of a set of twenty-one fundamental competencies specifically associated with differentiation. It has provided direction for data collection, analysis and interpretation. Each Test 1 item corresponds to one element of the framework, characterized by its *question type* and solution pathway involving a *process* (formulation, interpretation, translation, or combinations of them) in each *representation* (numerical, graphical and symbolic). (See page 3 for more detail.)

A second objective of both studies relates to teacher ″privileging″. Wertsch (1990) describes how different forms of mental functioning dominate in particular contexts and are influenced by a range of socio-cultural factors while Thomas, Tyrrell, and Bullock (1996) discuss the importance of a variety of teacher-related factors including attitude towards technology, and personal beliefs about the ways students should be taught. The first study showed that teacher ″privileging″ impacted on the ways students used the technology, and what they learnt. Teacher A ″privileged″ technology, symbolic algebra, and procedures for standard tasks; Teacher B, conceptual understanding, and by-hand algebra; and Teacher C, conceptual understanding and graphical methods. These privileging patterns resulted in different cognitive experiences and learning outcomes. Class A used computer algebra more frequently, and was more successful with symbolic items (less successful with conceptual items). While Classes B and C were more successful with conceptual items, Class B was more successful with by-hand algebra and Class C used graphical (non-calculus) methods more frequently as an alternative to symbolic procedures.

The purpose of this paper is to report some of the outcomes of the follow-up research project namely, the facility of each class with numerical, graphical and symbolic representations of derivative using test items based on the *derivative competency framework*. An attempt is also made to relate differences between the two classes to differences in the personal ″privileging″ of each teacher.

The teaching trial, research methods and data collection

In the repeat trial, Teachers A and B, at the same school (same identity as 1998) taught a slightly modified calculus program. (Teacher C from the 1998 trial did not participate). Both teachers helped revise the twenty-lesson introductory calculus program which was given a stronger emphasis on the concept of derivative in numerical, graphical and symbolic representations, and links between them, and a reduced emphasis on CAS. They subsequently taught it to Classes A and B, shared ideas about lessons, used a common teaching program and lesson notes they helped prepare, and gave students identical worksheets. Both teachers were experienced teaching with the TI-92 calculator. Prior to the trial, thirty-three Year 11 students (aged approximately 17 years) learnt how to use the TI-92 which they used optionally during the teaching program for all class work, homework and tests. The first author observed and audiotaped all the lessons, maintained a journal, interviewed the teachers individually before and after the program, and conducted task-based interviews with fifteen students. In contrast to the 1998 study where the ability levels of students in the three classes were normally distributed and evenly matched, the two classes had different distributions. School based assessments over two years (verified by a pre-test) showed that while both classes had a majority of weak students, Class A’s higher average attainment (including algebra and graphs) was due to the presence of more highly competent students. Students completed questionnaires, challenging assignment questions and completed two written tests. (See sample items in Table 1.)

The derivative competency framework

The *derivative competency framework* consists of a comprehensive (minimum) set of twenty-one competencies associated with the concept of differentiation and a broad view of its structure is evident in Table 3. Each Test 1 item matches one specific competency, characterized by its *question type* and the two aspects of the solution pathway, *process* and *representation*.

- *Question type*; numerical, (n), graphical (g), or symbolic (s)
- *Process* involved; formulation, interpretation, formulation and interpretation, translation and formulation, or translation of derivative
- *Representation*; numerical (N), graphical (G), or symbolic (S)

The *question type* is associated with the words of the question. ″Find a rate of change″ is classified as numerical (n); ″find a gradient″, graphical (g); and ″find the derivative″, symbolic (s).

P*rocesses* include; formulation (the cognitive ability to recognize that a derivative is required and know how to find it), interpretation (the ability to reason about a derivative in natural language), and translation (knowledge that a derivative determined in one representation, has meaning in a different representation).

R*epresentation* is governed by the method associated with finding the derivative. An approximate numerical (N) derivative at a point, is determined as a difference quotient (its limit found by investigation); a graphical (G) derivative is determined as the gradient of the tangent to the curve at a point; and a symbolic (S) derivative is determined as a function by directly manipulating formulae or as the limit of a function. These methods are described by Dick (1996).

Individual competencies and Test 1 items

Only Test 1 is reported in this paper. It was used to identify which differentiation competencies each student had mastered and consists of 21 items, (7 in each representation). Each item is designed to match one competency of the *derivative competency framework*. Five individual competencies are embodied in the five sample Test 1 questions displayed in Table 1 and discussed below.

Q.1 ″Find the derivative $f'(5)$″ is a symbolic (s) question which requires a numerical (N) formulation (decision to find an approximate derivative using a difference quotient). The competency is *(s to N) translation* & (*N) formulation.*

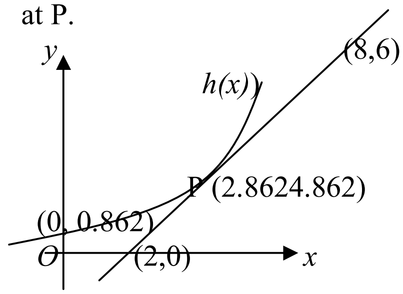
Q.2 ″Give the reason″ about a rate of change is a numerical (n) question, and requires reasoning about a numerical (N) derivative. The competency is (*N) interpretation.*

Q.3 ″Find derivative″ is a symbolic (s) question which requires a graphical (G) formulation (decision to find gradient of tangent at P). The competency is *(s to G) translation & (G) formulation*.

Q.4 ″Find the gradient″ is a graphical (g) question which requires a graphical (G) formulation (decision to find gradient of the tangent to the curve at x = 3). The competency is *(G) formulation*.

Q.5 ″Find the rate of change″ is a numerical (n) question which requires a symbolic (S) formulation (decision to find derivative using symbolic rule). The competency is (*n to S) translation* & (S) *formulation*.

Table 1. Sample Test 1 questions in each representation

	Representation	
Numerical (N)	**Graphical (G)**	**Symbolic (S)**
Q.1 The co-ordinates of points on a curve close to P where x = 5 are displayed in the table below. (Table is given). Find the best possible estimate of the derivative $f'(5)$. **Q.2** At 1.00pm, the rate of change of temperature in your house is +3 degrees Celsius (°C) per hour. Immediately after 1.00pm, is the temperature in the house most likely to: decrease, stay the same or increase. Give a reason for your answer.	**Q.3** The graph of the function h(x) is sketched below. The tangent at point P, on the curve h(x) has also been drawn. Find the value of the derivative of h(x) at P.  **Q.4** Use a graph of $y = x^2 + x - 10$ to find the gradient of the curve at x = 3.	**Q.5** The height of a plant can be determined by the formula $H(t) = 7t^3 - 3t^2$, where H is the height of the tree in metres, and t is the number of years since the tree was first planted. Find the rate of increase of the tree's height, 2 years after it was planted.

Results

Class facility

Competency is demonstrated by the student if the written solution on the test item indicates awareness of all of the necessary cognitive steps. The competency is not demonstrated if c*onceptual errors* are made, such as selecting an inappropriate representation, incorrect formulation (such as failure to choose two points on the tangent to the curve) or interpretation mistakes. However, competency may be demonstrated if *procedural errors* are made (such as algebraic, graphical, calculation, or careless mistakes e.g. transcription). Students from both classes used the calculator for numerical calculations, determination of a graphical derivative on Q.4 above, and only rarely for computer algebra.

On the twenty-one Test 1 items, the average number of competencies achieved by Class A (N = 14) students was 9.3 (St.D. 3.8), and Class B (N = 19), 9.7 (St.D. 3.4). C*lass facility* is defined as the percentage of attempts where the specified competency is demonstrated so Class A's facility is 44% and Class B's 46%. The fairly low facility of each class was not entirely unexpected as each class has a majority of lower ability students (indicated by prior school testing). However, the almost identical facility was unexpected since Class A had a higher proportion of highly competent students (discussed earlier).

Class facility in each representation

Class facility on the 7 items in each representation, is determined by a variety of factors including *attempt rate* (percentage of attempts made to solve the specified items), the percentage of *valid* attempts (where an appropriate representation was used), and percentage *success* (where competency was demonstrated on valid attempts). A student is considered to be *proficient* in the representation if he/she demonstrates competency on at least four of the seven items in each representation. Table 2 displays these two characteristics, ***percentage of proficient students*** and factors which determined overall ***class facility*** in each representation.

Table 2. Percentage of proficient students and factors which determine class facility in each class and representation

Factors influencing class facility	**Class**	**Representation** Numerical (N) (7 items)	Graphical (G) (7 items)	Symbolic (S) (7 items)
% of proficient students	**A** (N = 14)	29	50	43
	B (N = 19)	32	32	53
Attempt rate	**A** (N = 14)	84	87	83
	B (N = 19)	75	83	90
% of *valid* attempts (the representation used was appropriate)	**A** (N = 14)	78	80	81
	B (N = 19)	80	77	93
% *success* (competency was demonstrated on valid attempts)	**A** (N = 14)	54	68	70
	B (N = 19)	64	69	65
Class facility	**A** (N = 14)	36	48	47
	B (N = 19)	39	44	55

Symbolic representation: Class B has a higher proportion of symbolically proficient students (53%) and its superior symbolic facility (55%) is due to its higher attempt rate (90%), and on the (93%) of valid attempts, the representation was appropriate, affording an opportunity for success.

Graphical representation: Class A has a much higher proportion of graphically proficient students (50%) and its slightly superior graphical facility (48%) is due to its slightly higher attempt rate (87%) and marginally higher proportion of valid attempts (80%).

Numerical representation: The proportion of numerically proficient students is low in both classes (~30%). Class B's marginally better numerical facility (39%) is due to a higher success rate on valid attempts (64%) although Class A has a higher attempt rate (84%).

Class A's highest facility (with 50% of proficient students) is with the graphical representation (48%). This is almost identical its symbolic facility (47%) while its numerical facility is only 29%. In contrast, the highest facility for Class B (with 53% of proficient students) is the symbolic representation (55%) while its graphical (44%) and symbolic (32%) facilities are much lower.

Class facility in each representation sub-group

Each representation may be divided into three sub-groups, determined by the *question type* (described above). Table 3 shows the facility of each class (calculated as before) in each sub-group of competencies in each representation.

Table 3. Class facility for each representation sub-group on Test 1 items

	Class	**Representation** **Numerical (N)** (7 items)	**Graphical (G)** (7 items)	**Symbolic (S)** (7 items)
Numerical ***question type*** **(n)**		**nN** (3 items)	**nG** (2 items)	**nS** (2 items)
	A (N = 14)	48	14	50
	B (N = 19)	63	3	55
Graphical ***question type*** **(g)**		**gN** (2 items)	**gG** (3 items)	**gS** (2 items)
	A (N = 14)	36	60	43
	B (N = 19)	26	65	63
Symbolic ***question type*** **(s)**		**sN** (2 items)	**sG** (2 items)	**sS** (3 items)
	A (N = 14)	25	64	48
	B (N = 19)	16	55	49

Compared to Class B, Class A's slightly superior graphical facility (48%, Table 2) is due to better facility with two of three graphical sub-groups, sG (64%) and nG (14%) and which involve translation to the graphical representation.

Class B's superior symbolic facility (55%, Table 2) relative to Class A is largely due to its much better gS (63%) and marginally better nS (55%) facility both of which involve translation to the symbolic representation.

Class B's marginally better numerical facility (39%, Table 2) is due to its better nN (63%) facility. Class A has better gN (36%) and sN (25%) facility, both of which involve translation to the numerical representation.

Class performance, teacher influences and CAS

The first author carefully observed every lesson and witnessed the different teaching practices of each teacher and learning outcomes of each class. Both teachers used CAS to demonstrate the planned conceptual activities (such as dynamic demonstrations to show the relationship between gradient of secant, curve and tangent to curve). Teacher A stressed routine procedures for routine numerical, and graphical tasks, emphasized algebraic routines and use of computer algebra, often used the calculator in front of the class, and permitted the students free use of the calculator. Class A developed equal facility with both graphical and symbolic representation, and compared to Class B, a higher proportion of graphically proficient students (Table 2) and better computer algebra capabilities (evident on Test 2) but of no benefit on the conceptual Test 1. Teacher A taught Class A students to respond to specific data cues and to use the representation suggested by the context of the question which often resulted in a translation to the graphical or numerical representation. As a consequence of Teacher A's emphasis on standard procedures and use of technology, Class A developed better facility with competencies involving translation to the graphical (and to the numerical) representation (Table 3) and in consequence a slightly superior graphical facility.

Teacher B taught for conceptual understanding and encouraged his students to develop their intuitive ideas about rate of change, slope and the limit concept. This contributed to Class B's superior facility with the nN sub-group (Table 3) which involves interpretation and formulation competencies, and a consequent slightly superior facility with the numerical representation. Teacher B consistently and deliberately stressed the importance of the symbolic derivative and Class B developed a higher proficiency with the symbolic representation than with the other two representations. In addition, compared to Class A, Class B had higher proportion of symbolically proficient students, attempted symbolic items more frequently, and made more valid selections (Table 2). Teacher B linked the numerical and graphical derivatives to the symbolic derivative by illustrating symbolic ideas with graphs and enactive representations (such as physical arm movements) to illustrate slopes of tangent lines. Class B's superior facility with symbolic representation, in particular competencies that involved translation to the symbolic representation, is a direct consequence of Teacher B's focus on teaching for conceptual understanding and use of the symbolic representation (including by-hand algebra). He allowed the students to use the calculator to determine graphical and numerical derivatives and for inducing the symbolic rules but otherwise insisted that students use by-hand algebra.

Discussion

Class A and Class B achieved an almost identical average number of differentiation competencies on Test 1 yet their facility in each representation and sub-group were different. Teachers A and B intended to teach the same material in the same way, however they taught with different personal interpretations and emphases. Both teachers' privileging patterns were identical to those in the first study. Once again, Teacher A ″privileged″ procedures for standard tasks, and use of technology including computer algebra, and Teacher B, conceptual understanding, the symbolic representation and by-hand algebra. In consequence, Class A and Class B students had different cognitive experiences and acquired different differentiation competencies which related directly to the ways they were taught. Class A students used technology more frequently, relied on standard procedures and developed both symbolic and graphical facilities to the same extent. Compared to Class B, Class A's slightly superior proficiency with the graphical representation particularly translation to the graphical (and to the numerical) representation, developed through greater use of the more intuitive, and visual graphical representation of derivative and use of context cues in the questions. In contrast, Class B students preferred and were more competent with the symbolic representation, particularly translation to the symbolic representation, significant in light of their weaker algebraic background and lower school performances. In an interview, Teacher B stated that he strongly directed his students towards the symbolic representation because of his personal beliefs that the symbolic representation was the most important, and that by-hand algebra was crucial for understanding. He was also convinced that his less able cohort of students would not cope successfully with more than one representation.

How important is by-hand practice for these essentially conceptual items? Lagrange (1999b) suggests that ″techniques″ play an important role in conceptual understanding. Techniques involve both the conceptualization of the steps required and their execution. Class B students used CAS for learning about the numerical, graphical and symbolic derivatives but not for executing algebra, yet they achieved higher facility on symbolic items. Although CAS use was optional, it was not needed or used on the test as the algebraic demands of items were simple. Is Class B's greater symbolic facility (in spite of poorer performance on school tests including algebra) due in part to Teacher B's insistence on by-hand practice?

During these lessons, the two classes achieved similarly overall but there were significant differences, which classroom observations linked with their teacher's ″privileging″. Is it possible that in the future, teachers using CAS will be able to identify their own ″privileging″ and modify their teaching strategies so that all students in their classes will have the opportunity to acquire the complete set of numerical, graphical and symbolic competencies associated with the concept of derivative? Was twenty lessons sufficient time for students to develop all competencies and does student aptitude and preference for representation place a natural limit on what can be achieved?

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