Priorities for Middle School Mathematics: New Directions from Research for Number and Algebra

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This article highlights some fundamental areas of middle school mathematics that I believe are priorities for attention in the middle years of schooling. In recent years, we have conducted several research projects at the University of Melbourne on number and algebra learning. Examining the work of several thousand children has shown that difficulties in certain topics persist to affect many areas of mathematics. Two such topics that consequently need special attention are discussed in this article.

Firstly, I discuss the need to develop strong conceptions of number, not just of positive whole number. Secondly some aspects of making a transition from arithmetic to algebraic thinking are discussed. Arithmetic leads to algebra, but with a twist! Both topics require students to make fundamental re-organisations of their thinking: helping children do this is a main agenda of mathematics in the middle years on which success in later years depends.

Developing strong conceptions of number, beyond whole numbers.

The first concept of number is of positive whole numbers and one of the tasks of upper primary school is to extend children's thinking so that decimals, fractions (and negative numbers) are considered as "proper" numbers. There are many examples of children unable to solve problems because they do not look beyond whole numbers and many examples of children who believe that like the whole numbers, the decimal and fraction number systems are discrete (with gaps in between them). The number line is not well established (e.g. many students put fractions between 0 and 1 on the "negative" side).

There is confusion about what is and is not a number. For example when one number is divided by another, teachers know the answer is a number. But the first way that children learn division is with remainder: 17 divided by 3 is 5 remainder 2. Is this a number? Children who do not move on to seeing the answer instead in terms of a fraction or a decimal are disadvantaged in many ways, such as in developing a concept of inverse operations. For teachers, division is the inverse operation of multiplication because

$$(a/b) \times b = a \text{ and } a \times b/b = a.$$

However, whilst a child who can only think of whole number remainder division may agree that $(17 \times 3)/3 = 17$ he or she will have difficulty making any sense of $(17/3) \times 3$: what is the answer when 5 remainder 2 is multiplied by 3? Is 15 remainder 6 equal to 17?

Evidence gathered around the world has pointed to the extent of the change that students have to undergo to properly understand the decimal number system. Recent TIMSS data shows that Australian children are above the international average, but still many have difficulty with apparently simple tasks. For example, in the item shown in Figure 1, 47% of Australian 13 year olds correctly selected the second list as showing smallest to largest (data held at ACER). Internationally 44% of students were correct, although 84% of Singapore students were correct. On the other hand, a quarter of Australian children selected the first list in Figure 1 as showing the numbers from smallest to largest. In this list, if you interpret the decimal point as a fraction vinculum instead (so it becomes 1/345, 1/19, 1/8, 1/5) then the numbers are indeed from smallest to largest. Our research in Victorian schools indicates that about 10% of children from Grade 6 to Year 10 consistently make judgements like this, even in much simpler situations. We have carefully analysed the types of thinking that might lead to behaviour like this (there are several) and are creating a website (currently at http://online.edfac.unimelb.edu.au/485129/DecProj/index.htm) for use by student teachers and practising teachers which explains some of the ideas involved and offers teaching ideas to address this and similar problems.

As is typical of learning maths, the path to mastering decimal numbers is not smooth. Just when students learn how decimal numbers work and realise that decimals such as 3.1, 3.10 and 3.100 are all equal, they need to learn that this is only the case for the numbers of pure mathematics. When these are the results of measurement instead, then 3.1, 3.10 and 3.100 are no longer the same. As a measurement, 3.100 indicates a much more precise measurement than 3.1 and the final zeros really make a difference.

In this way, learning decimals is typical of learning many mathematics topics. There is a complex web of inter-related ideas. Previous learning will both support and confuse but there are plenty of things teachers and parents can to do help. And just when one idea is mastered, there is another twist.

Which list shows smallest to largest?

0.345	**0.19**	**0.8**	**1/5**
0.19	**1/5**	**0.345**	**0.8**
0.8	**0.19**	**1/5**	**0.345**
1/5	**0.8**	**0.345**	**0.19**

Figure 1: One item from the TIMSS study for 13/14 yr olds.

Making the transition from arithmetic to algebra.

Our recent research studies on algebra have involved over 1000 students from secondary schools of all types. Even on easy problems, the overall success rate of students using algebra to solve problems is low, although there is great variation between schools. Many students use no algebra, instead relying on logical arithmetic reasoning (which can be very difficult) or guess and check, a useful but not-very-powerful method. For simple problems such as MARK and JAN (see Figure 2, where typical student solutions are shown), both of these methods are quite adequate. A guess and check solution (Solution 1) is quite easy, especially as the answer does not involve non-integral numbers. Two different types of solutions by logical arithmetic reasoning are available (Solutions 2 and 3) and for this question both are quite simple. However, when the problem is a little more difficult (see, for example, MARK, JAN and BERNIE in Figure 3) non-algebraic methods become much more difficult. Guess and check is less straightforward, as the answer is not an integer. Logical arithmetic reasoning generalising the method of Solution 2 in Figure 2 is extremely hard and even generalising the method of Solution 3 is challenging for some students. In this case, the power of algebra is needed but many students were unable to change their approach.

By examining the work of many students, we have come to see that difficulties for students arise from not making the transition from arithmetic thinking to algebraic thinking. Figure 4 lists some of the ways in which arithmetic thinking and algebraic thinking differ. Differences arise in the idea of the unknown (whether it is transient or fixed) and in the basic approach to problem solving. Algebra provides a very different method, not just a new language.

MARK AND JAN

Mark and Jan share \$47, but Mark gets \$5 more than Jan. How much do they each get?

SOLUTION 1 Guess and check (Year 9)

> 15 + 32 = 47 but the difference is not \$5
> 16 + 31 = 47 but the difference is not \$5
> ……………
> 21 + 26 = 47 and the difference is \$5.

SOLUTION 2 Logical arithmetic reasoning (Year 9)

> $47/2 = 23.5 - 2.5 = x$
> $47/2 = 23.5 + 2.5 = y$

SOLUTION 3 Logical arithmetic reasoning (Year 10)

> $y = (47 - 5)/2 + 5 = 42/2 + 5 = 26$
> $x = (47 - 5\ /2 = 42/2 = 21$

SOLUTION 4 Algebraic "do the same to both sides" solution (Year 11)

> $x + (x + 5) = 47$
> $2 \times x + 5 = 47$
> $2 \times x = 42$
> $x = 21$

Figure 2: The problem MARK AND JAN and four solutions from students in Years 9 to 11.

MARK, JAN AND BERNIE

Mark, Jan and Bernie share \$47. Mark gets \$5 more than Jan and \$10 more than Bernie. How much do they each get?

Figure 3: Mark, Jan and Burnie

Arithmetic Thinking	Algebraic Thinking
Work from knowns to unknowns	Work on unknowns
Unknowns transient	Fixed unknowns
Equation as a formula to produce answers	Equation as a description of the situation
Chains of successive calculations	Chains of logically linked equalities

Figure 4: Contrasting arithmetic and algebraic thinking

Solutions 2 and 3 of MARK and JAN illustrate how arithmetic solutions to a problem work from knowns to unknowns, by a process of calculation. For example, from the known numbers 47 and 5, one finds the new quantity $42 (the amount of money that is to be shared equally in Solution 2 after the extra $5 has been given to Mark) and then from this new known, one finds $21 (the amount that they both get). The unknowns are transient: first the aim is to find one, by knowing that one we can find the next, and so on in a chain of successive calculations. In contrast, the algebraic solution is quite different. Instead of immediately progressing towards a solution, one somewhat passively describes the situation: $x + (x + 5) = 47$. Many students to whom we have spoken do not feel that this is a helpful thing to do. Then, instead of "actively" working towards a solution by calculation, one then works on the whole equality, producing more equations until eventually the solution appears, also as an equation: $x = 21$.

The extract from our interview with Les below when he is working on MARK AND JAN, illustrates how he uses algebraic notation in an "arithmetic" way, as a way of saying something about the various "unknowns" that he encounters as he solves the problem by logical arithmetic reasoning. He uses notation to informally track his thinking. Les begins by writing $5 + x = 47$ and explains it thus:

L: x is what is left out of $47 if you take 5 off it.
I: What might the x be?
L: Say she gets $22 and he gets $27. They are sharing two x's.
I: What are the two x's?
L: Unknowns…they are two different numbers, 22 and 27.
I: So what is this x? (pointing to $5 + x = 47$)
L: That was what was left over from $47, so its $42.

Conclusion

These few examples indicate some of the ways in which arithmetic leads to algebra, but with a twist. A strong understanding of number and number operations is essential for learning algebra, but there are marked transformations which have to occur in students' thinking to become comfortable with problem solving using algebra. As research highlights how students' thinking has to change, there is an opportunity to revitalise mathematics in the middle years of schooling. Now that calculators have lightened considerably the burden of teaching algorithmic mathematics that especially dominated the middle years, there is an opportunity to rethink fundamental aspects of number, number operations and the transitions that students have to make to becoming confident mathematicians.

Further reading

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