

A FLYING START TO ALGEBRA

Algebra learning has its roots in the early grades, when children notice regularities in the ways that numbers work. This is recognized in the *Curriculum and Evaluation Standards for School Mathematics* (1989) which states that in Grades K-4 children should be encouraged to

- recognize, describe, extend, and create a wide variety of patterns;
- represent and describe mathematical relationships.

From Grade 5, according to the *Standards*, students should

- extend their understanding of whole number operations to fractions, decimals, integers, and rational numbers;
- understand how the basic arithmetic operations are related to one another.

We have conducted research over several years with middle school and high school students learning algebra (references are given at the end of this article). The results consistently highlight that students find algebra hard to learn unless they have a good knowledge of number properties and basic operations. They need to have thought about the general effects of operations on numbers, and not just focussed their attention on getting answers to computations. As well, they need to feel comfortable using large numbers, fractions and decimals, so that they can recognize when a general relationship or rule applies to the whole range of numbers. Children's discoveries about how numbers work form the building blocks of generalised number knowledge, which is developed and expressed in later years as algebra.

This article concentrates on how number work in the elementary school can be extended to prepare students for algebra. We suggest some practical strategies for teachers to work on several problem sites for learning algebra:

- understanding equality
- recognising the operations
- using a wide range of numbers
- understanding important properties of numbers
- describing patterns and functions.

Teachers can incorporate these strategies into their existing programs. For example, it is often worthwhile to spend time on variants of one problem, encouraging children to find out what happens when different

numbers are used or when a pattern is extended. Teachers will find many more ways in which the K-6 curriculum can be enriched to prepare students for algebra in the February 1997 focus issue of *Teaching Children Mathematics* on algebraic thinking.

UNDERSTANDING EQUALITY RELATIONSHIPS

The problem

The language of arithmetic is focussed on answers. The language of algebra is focussed on relationships. For example, compare the arithmetic statement

$$287 + 146 = 433$$

with a typical algebraic statement

$$2(x + 1) = 2x + 2$$

The arithmetic statement gives an answer and the "=" sign indicates that this answer has been found. While they are learning arithmetic, children come to interpret the "=" sign as meaning "work out the answer". When they see $4 + 7 =$, for example, they believe they should write the answer, 11. They see the left hand side of the equation as the "question", and the right hand side as the "answer" (which in their experience is a single number). This interpretation, if it persists, is a major obstacle to understanding how algebra is used.

In the algebraic statement $2(x + 1) = 2x + 2$, although $2(x + 1)$ can be interpreted by students as a "question", $2x + 2$ does not look like an "answer". Instead, we have a statement about equality; the two quantities are equal and one of them can replace the other when required. Students have to learn how to write down equalities and how to generate chains of logically equivalent statements of equality such as

$$\begin{array}{ll} \text{if} & (8x - 3) \div 3 = 2x \\ \text{then} & 8x - 3 = 6x \\ \text{and so} & 8x - 6x - 3 = 6x - 6x \\ \text{so} & 2x - 3 = 0 \\ \text{so} & 2x = 3 \\ \text{and so} & x = 1\frac{1}{2} \end{array}$$

Beginning algebra students often find it hard to deal with this form of equality and how one equality follows from another. It will be helpful

if they have already seen and used statements of equality in arithmetic.

Something for teachers to think about is what to do when students use the "equals" sign to link steps in a computation. When students write, for example, $5 + 23 = 28 \times 3 = 84$, they are writing exactly what they do on their calculators. So how can it be wrong? The calculator interprets the sign as "carry out the operation", but in formal mathematics the $=$ sign links two expressions that have exactly the same value. Some teachers may choose to explain these two different meanings and have students use another sign - perhaps one they have invented themselves - to link steps in their working. For example, they could use an arrow to mean "gives" in $5 + 23 \rightarrow 28 \times 3 \rightarrow 84$: five plus twenty three gives 28, which multiplied by 3 gives 84.

Some suggestions

- Often write and have children write statements of equality with the "answer" first. For example, instead of $5 + 9 = \square$, sometimes try $\square = 5 + 9$.

- Use a geoboard and a piece of string 24 geoboard units long to make different shaped rectangles. Children will see that all their rectangles have the same perimeter (24 units). They may also notice that the areas of the rectangles are different, although the perimeters are the same. The important focus of this activity should be the writing of many different statements of equality. For example,

$$24 = 2 + 10 + 2 + 10$$

$$24 = 4 \times 6 = 6 + 6 + 6 + 6 = 12 + 12$$

$$3 + 9 + 3 + 9 = 4 + 8 + 4 + 8$$

and so on.

Seeing the reasons behind relationships such as the last one requires generalisation about properties of numbers that is deeply algebraic. Referring back to the fixed length of the string will help children see the reasons behind it. Erna Yackel's article in the February 1997 issue of *Teaching Children Mathematics* provides similar activities and a discussion of their value for developing algebraic thinking (Yackel 1997).

To make this activity more challenging for older students, the

units on the board can be given a value other than 1. Try 0.2, or $\frac{1}{2}$, for example.

- • Have children write their answers in alternative ways. For example, instead of $5 + 9 = 14$, they can write $5 + 9 = 10 + 4$ or $5 + 9 = 2 \times 7$, and so on. From the earliest years, children can appreciate that there are many ways of expressing a number. Teachers should encourage them to demonstrate this understanding. Figure 1 shows the work of two first-graders, David and Amy, when their class was asked to draw Santa's eight reindeer and count the legs. They were asked to write a number sentence to show how they had counted, and then write other number sentences to show other ways. The statements in inverted commas in Figure 1 were put there by the teacher to record what the children said. Note that although Amy says that she worked out her number sentence with a calculator, this is not how she discovered it. She grouped the legs in different ways - ones, two, threes and fours - counted them, and then checked with the calculator.

Figure 1 about here

RECOGNISING THE OPERATIONS

The problem

The *Standards* states (p. 91) that students in Grades 5-8 should have the opportunity to "understand how the basic arithmetic operations are related to one another". This understanding develops slowly and depends on first recognising the four operations and why they are different. To solve many problems, in or out of school, it's often not important to know what operations are being used. In everyday life we frequently use methods based on counting, adding, doubling and halving, that can be carried out mentally. We break down a complex problem into small steps so that the solution becomes a collection of simple computations. In contrast, to use algebra for solving a problem, the focus of attention is not on getting numerical answers to each step of the solution, but on the operations used. Relationships and procedures must be recognised and made explicit. It is important therefore that students get experience in identifying which operation they are using

to solve a problem. As stated in the *Standards* (p. 42), "connecting problem structures to operations should be emphasized throughout Grades K-4 for both one-step and appropriate two-step problems".

One of the biggest obstacles to algebra learning in the middle years is a limited understanding of multiplication and division. Students who see multiplying only in terms of repeated adding can think of 3×4 as $4 + 4 + 4$, but are not able to make sense of $m \times 4$ because it doesn't tell you how many 4's to add. Similarly, division needs to be understood as an operation in its own right as well as being related to repeated subtraction. Children can be given problems that move them forward from using only primitive methods based on counting, adding and taking away. They need to become familiar with the concepts of multiplication and division that they will need for algebra.

Some suggestions

- Figure 2 shows four ways to solve the CAGES problem, each method based on one of the four operations. Children should understand all these methods. It is important that they use them in a variety of contexts, and learn to choose which one is most suitable for a particular problem. Whereas numerically addition, subtraction, multiplication or division can be used, in algebra only one operation (division) will do. If there are N birds, and x to each cage, the number of cages is $\frac{N}{x}$.
- To solve the MIGRATING BIRDS problems (see Fig. 3), children will see that the first problem is easily solved without thinking about division, but for the second problem division is much quicker than repeated addition or subtraction. For both problems, multiplication should be used to check the answers.

Figure 2 about here

Figure 3 about here

USING A WIDE RANGE OF NUMBERS

The problem

Some students enter the middle years with very limited experience of numbers other than small whole numbers. They are not sure whether the procedures they have learned to use with these numbers can also be used for large numbers, decimal numbers and fractions. This uncertainty is not surprising, because students learn to do written computations differently for different kinds of numbers. For example, adding up fractions is not like adding up decimals, as shown below.

$$\text{Fractions} \quad \frac{3}{4} + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{3+2}{4} = \frac{5}{4}$$

$$\begin{array}{r} \text{Decimals} \quad \quad \quad 0.75 \\ + \quad \quad \quad \underline{0.50} \\ \hline 1.25 \end{array}$$

When we talked with 14-year-old students while they worked on some algebra tasks, we noticed that they expected questions containing decimal numbers to be much harder than whole-number questions, or to require different rules. We had asked them translate some statements into algebraic language. One statement was:

If y is multiplied by 8, the result is 24.

Another was:

Three times y is equal to 0.051.

Students found the first question easy. They did not hesitate in writing correct equations such as $8 \times y = 24$, or $8y = 24$. However some of them were worried by the second question. They took much longer to translate it than they had taken for the whole-number question, although it has the same structure ($3 \times y = 0.051$ or $3y = 0.051$). Some said they had no idea what to do. Maria said immediately, "I'm not good at decimals". Larry looked at the question and said, "That number! I don't like those numbers, decimals." Ben said, "I don't think I know how to do this one".

Students like Maria, Larry and Ben need to know that the operations they carry out on small whole numbers work for all numbers. Unfortunately some teachers, and many textbooks, tend to restrict

students' experiences to the use of small whole numbers, instead of using the whole range of numbers. In the past, this could be understood because of the difficulty of pencil-and-paper computation. However now that students have access to hand-held calculators, there is no reason to restrict so tightly the range of numbers they use.

Some suggestions

- Use large numbers in problems. For example move children on from "three people share 15 apples" to "three people share \$42" to "three chickens weigh 1710 grams" and "400 ibises eat 90000 grasshoppers". Variants of one problem will help children recognise that the same operation applies to each, regardless of the size of the numbers. Use calculators when necessary, so that attention is focussed on the operation and not on the numbers.
- Number patterns can help develop ideas of decimals (see Fig. 4). Too often, pattern work is restricted only to whole numbers. Patterns which apply to whole numbers can be investigated to see if they work with decimals, fractions and negative numbers. Sometimes, there will be interesting reasons why they do not work. One example (when a student sees the pattern as adding a 0 rather than multiplying by ten) is given later in this paper. Another interesting example is given in the article Liz's Pattern (Land & Becher 1997) in the focus issue of *Teaching Children Mathematics*.

Figure 4 about here

- Take opportunities to generalise mathematical ideas beyond whole numbers. Also in the 1997 focus issue of *Teaching Children Mathematics* there is a description (Lubinski & Otto 1997) of how a first grade class extended the mathematical principles involved in the story *The Doorbell Rang* (Hutchins 1986) beyond whole numbers to halves and quarters. As they continued to work with the situation of sharing 24 cookies with more and more children, the class began to understand the inverse multiplicative relationship linking the variables the number of children and the amount of cookies that each could receive. The teachers questioning kept the focus on the quantities involved and their relationships, rather than just on the numbers involved.

UNDERSTANDING PROPERTIES OF NUMBERS

The problem

One of the most frequent algebraic acts is renaming expressions. For example,

$$a - (b - c) = (a - b) + c$$

In order to understand what is going on here, students need to know how subtraction behaves. If they have forgotten the rules for operating with algebraic expressions, they should be able to go back to arithmetic and see what happens in specific instances with numbers instead of letters. But very often their knowledge of number properties is not strong enough. Children need to strengthen their intuitive knowledge of number properties, so that they know, for example, "If you subtract 1 less, then the answer will be 1 more". They need to feel confident, without checking by computing both sides, that

$$3037 - (258 - 1) = (3037 - 258) + 1$$

so that they can later generalize to the algebraic statement

$$a - (b - c) = (a - b) + c$$

Activities such as those shown in Figure 5 will help to develop this knowledge.

Children who have a weak understanding of number properties are often not concerned about the effects of re-grouping or re-ordering numbers. Even in the middle years, some students are not aware that it matters how numbers are grouped or which operation they do first. For example, 14-year-olds working on a problem that required them to work out $3 \times 5 + 9 \div 3$ came up with three different answers. In Figure 6 we illustrate how a problem like this can be used for discussing with students why grouping and ordering are important and how brackets are used to show which interpretation is intended.

Figure 5 about here

Figure 6 about here

DESCRIBING PATTERNS AND FUNCTIONS

The problem

Working with middle school students, we have found that they often see relationships in patterns which are valid but cannot be expressed in mathematical symbols. For example, for the following table,

x	1	2	3	4	5	6	7	8	...
y	9	19	29	39	49	59	69	79	...

some students noticed the patterns in each row - "x goes up by ones and y goes up by tens" (a few students said "y goes up by 9's", since they counted how many digits were missing in the gap between 9 and 19). Some noticed that y increases at a faster rate than x - "For every x, there are ten y's". Some noticed a link between each x and the y to its right below - "The x tells you what the tens digit is in the next y". Some found rules linking x and y - "To work out y, put a zero onto x and take 1" or "Take 1 off the x, and write 9 after it". None of these descriptions can be written in algebra, although the students were able to use them to continue the table.

Students need opportunities to talk about the patterns they see and the ways to describe them mathematically. The student who said, "Put a zero onto x" needs to know that the operation involved is "Multiply x by 10". The students who thought that the difference between 9 and 19 is 9, because when they counted up they missed 9 digits, need to refine their understanding of "difference between" in terms of addition and subtraction, moving beyond counting.

Some suggestions

- In the early grades, children should become familiar with the many forms in which simple mathematical relationships can be expressed. For example, if $A = 30$ and $B = 15$, we can say "A is twice B", "B is half of A", "B doubled gives A", "A is 15 more than B", "B is 15 less than A", "The difference between A and B is 15". Children need to hear all these forms used by their teachers and their classmates, articulate them themselves, and know how one form can be changed to another.
- Encourage them to recognize both the successor and the functional relationship in a sequence such as 3, 6, 9, 12, 15, 18, 21, ... As well as "keep adding 3", they need to see the pattern of the three times tables 1×3 , 2×3 , 3×3 , 4×3 , ... and

recognize the operation "multiply by 3".

- Encourage them to look for a functional relationship between two variables, instead of just looking at changes in one variable. For example, use tables showing input and output numbers that are not arranged sequentially. An example is shown in Figure 7. Willoughby (1997) outlines a gradual approach to this across the grades K - 6.

Figure 7 about here

CONCLUSION

Algebra is that part of mathematical language that has been designed to express general relationships between numbers. To learn algebra, students need number knowledge that goes far beyond arithmetic calculations and "basic skills". In particular, they need to understand general properties both of numbers and of the operations that combine them. The activities we have described in this article will provide teachers with ideas for giving their students a flying start to algebra.

NOTE. Students may want to find out some of the amazing facts about migratory birds. The ones we refer to in the MIGRATING BIRDS problem are the red-necked stint *Calidris ruficollis* and the sharp-tailed sandpiper *Calidris acuminata*. Many students in North America will have seen the Canada goose, a much larger bird than stints and sandpipers. Where does its journey begin and end? How fast does it fly?

REFERENCES AND FURTHER READING

Hutchins, Pat. *The Doorbell Rang*. New York: Greenwillow Books, 1986.

Land, Jill E. & Becher, Paul G. "Liz's Pattern" *Teaching Children Mathematics* Vol 3 No. 6 (February 1997): 301 -304.

Lubinski, Cheryl A. & Otto, Albert D. " Literature and Algebraic Reasoning" *Teaching Children Mathematics* Vol 3 No. 6 (February 1997): 290 - 295.

MacGregor, Mollie, and Stacey, Kaye. "Backtracking, brackets, BOMDAS and BODMAS." *Australian Mathematics Teacher* 51 No.3 (1995): 28-31.

MacGregor, Mollie, and Stacey, Kaye. "Students' understanding of algebraic notation: 11-15." *Educational Studies in Mathematics* (in press)

MacGregor, Mollie, and Stacey, Kaye. "What is x?" *Australian Mathematics Teacher* 49, No.4 (1993): 28-30.

National Council of Teachers of Mathematics. *Curriculum and evaluation standards for school mathematics*. Reston, Va.: The Council, 1989.

Stacey, Kaye, and MacGregor, Mollie. "Building foundations for algebra." *Mathematics Teaching in the Middle School* Vol 2 No. 4 (February 1997): 252-260.

Stacey, Kaye, and MacGregor, Mollie. "Ideas about symbolism that students bring to algebra." *Mathematics Teacher* Vol 90 No. 2 (February 1997): 110 - 113.

Willoughby, Stephen S. "Functions from Kindergarten through Sixth Grade". *Teaching Children Mathematics* Vol 3 No. 6 (February 1997): 314 - 318.

Yackel, Erna. "A foundation for algebraic reasoning in the early grades." *Teaching Children Mathematics* Vol 3 No. 6 (February

1997): 276 - 280.

CAGES

John needs new cages for his 96 canaries. He will put 6 canaries in each cage. How many cages does he need?

Repeated Addition (use calculator)

$\boxed{6} + \boxed{6} = \boxed{=} \boxed{=} \boxed{=} \dots$ *I added six 16 times.*

Repeated Subtraction (use calculator)

$\boxed{9} \boxed{6} - \boxed{6} = \boxed{=} \boxed{=} \boxed{=} \dots$ *I subtracted six 16 times.*

Trial Multiplication and Addition (mental reasoning)

I know that ten 6's are 60, and five 6's are 30, $60+30=90$, that's 15. It's almost enough. We need one more cage. That makes 16 cages.

Division (use calculator)

$\boxed{9} \boxed{6} \div \boxed{6} = \boxed{=}$ *The answer is 16.*

Check that the answer "16 cages" is correct

Use a calculator to multiply 16 by 6. $16 \times 6 = 96$. *This equation tells me that if there are 16 cages, and each cage holds 6 canaries, there are 96 canaries altogether.*

Figure 2. A problem that can be solved by using addition, subtraction, multiplication, or division.

MIGRATING BIRDS

1. Every year when the weather in the Northern Hemisphere starts to get cold, small wading birds called stints and sandpipers set out from Alaska to fly to Australia. They have a long way to go, about 12000 miles. Suppose they travel 300 miles every day. How many days will the whole journey take?

Multiplying and Doubling (mentally)

In one day they go 300 miles, so in 10 days they go 3000 miles. Doubling that gives 20 days, 6000 miles. Doubling again gives 40 days, 12000 miles.

Division (mentally, or with calculator)

Divide 12000 by 300. $12000 \div 300 = 40$.

2. Some birds making the journey from Alaska to Australia take a longer route, about 13500 miles. If one of these birds flies 230 miles every day, how many days will the whole journey take?

Figure 3. Two division problems

EXTENDING A PATTERN

Use a calculator to find the output numbers when the rule is "Divide by 10".

Input number	Output number
50000
5000	...
500	..
50	5
5	0.5
0.5	...
0.05
0.005

What do you notice? Predict what happens if the rule is "Divide by 100". Check answers on your calculator. Predict what happens if you start with a number that is not a multiple of 10 (try 98765 for example) What is the smallest number you can get on your calculator?

Figure 4. Extending a number pattern to include decimals

TAKING AWAY MORE, GETTING LESS

1. Work this out. Use a calculator if you want to.

$$\begin{array}{r} 3037 \\ - \underline{258} \\ \hline \end{array}$$

Now write down the answers to these subtractions, without working them out. Explain how you know the answers are right.

$$\begin{array}{r} 3037 \\ - \underline{259} \\ \hline \end{array} \quad \begin{array}{r} 3037 \\ - \underline{257} \\ \hline \end{array} \quad \begin{array}{r} 3037 \\ - \underline{268} \\ \hline \end{array} \quad \begin{array}{r} 3037 \\ - \underline{358} \\ \hline \end{array} \quad \begin{array}{r} 3037 \\ - \underline{158} \\ \hline \end{array} \quad \begin{array}{r} 3037 \\ - \underline{288} \\ \hline \end{array}$$

2. Work this out.

$$\begin{array}{r} 6214 \\ - \underline{1989} \\ \hline \end{array}$$

Now write down six other subtractions that would also have this answer. Explain how you changed the numbers 6214 and 1989.

$$\begin{array}{r} - \underline{\quad} \\ \hline \end{array} \quad \begin{array}{r} - \underline{\quad} \\ \hline \end{array} \quad \begin{array}{r} - \underline{\quad} \\ \hline \end{array} \quad \begin{array}{r} - \underline{\quad} \\ \hline \end{array} \quad \begin{array}{r} - \underline{\quad} \\ \hline \end{array} \quad \begin{array}{r} - \underline{\quad} \\ \hline \end{array}$$

Figure 5. Developing confidence in understanding number properties

GROUPING AND ORDERING

1. Carla, Dan and Ella were working out what the answer is if you start with 3, multiply by 5, add 9 and divide by 3. They wrote the question down as $3 \times 5 + 9 \div 3$. They got different answers.

Carla said, "I know 3×5 is 15, and I know $9 \div 3$ is 3, so it's 15 plus 3. The answer is 18".

Dan said, "No, that's wrong. There's 3 at the front and then $\div 3$ at the end. The 3's cancel out, so you've just got 5 plus 9, that's 14".

Ella said, "I think the answer is 8 because you have to do it in order. Three times 5 is 15, then add 9 and you get 24, and then divide by 3 and you get 8".

Is Carla right? Is Dan right? Is Ella right?

2. How many different answers can you get for each of these just by adding brackets?

(i) $6 \times 12 \times 3$ (ii) $6 \times 12 \div 3$ (iv) $6 \times 12 - 3$ (iii) $6 \times 12 + 3$

What do you notice?

Try some other sets of three numbers.

What happens if you use fractions? Try $6 \times \frac{1}{3} \times 3$ and $6 \times \frac{1}{3} + 3$.

Figure 6. Three answers to one problem!

WHAT IS THE RULE?

Input number	2	7	9	5	10	6	20	100
Output number	6	11	13	9	14

What is the rule connecting input and output? Use your rule to work out the missing numbers in the output row. What will the output be if the input number is 2.5? 0.7? 3999?

Figure 7. Specifying a functional relationship as a general rule

Figure 1.