Mathematics Education for National Development

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Societies and governments around the world recognise the importance of mathematics for national development. Over this century, the fabric of society has become more and more underpinned by mathematical ideas. As a result, a major development in mathematics education in this century has been the increased amount of mathematics that all citizens are expected to know. At the beginning of this century, adequately educated people may have learned only arithmetic and measurement, but now a basic mathematics education includes ideas from algebra, geometry and statistics as well. In addition, the mathematics that is required for modern technology has greatly expanded. Technological leaders need a mathematics education that takes into account both the new uses of mathematics in science and technology and the new ways in which mathematics can be done with information technology.

In this address, I will use *coding theory* as one example of an area where new mathematical ideas directly influence technological development. The example shows the close links between advances in science and technology and modern mathematics. The digital revolution has "digits", numbers, the core of mathematics, at its heart. We will see how important advances from digital technology, such as the clarity of a digital phone call or of music from a compact disc depend on being able to calculate with the digits that carry information. I hope that all the groups in the audience, mathematics educators, science educators and those interested in technical education will find something interesting in coding theory itself. Furthermore, many implications for education can be drawn from this example. Time permits us to discuss just two:

- something unprecedented: the curriculum challenge put out by the power of information technology to "do mathematics", at least in its routine aspects
- something perennial: the pedagogical problem that arises from the deep abstractness of mathematics, from which it derives its power. Although it is abstract, it must always feel concrete.

What makes the digital revolution so revolutionary?

Why is it that the products of the digital revolution are so much better than the products that came before? Why, for example, does music from a compact disc sound much clearer than music from an audiotape? Why are long distance telephone calls much clearer now than they were in the past? How can such a huge amount of information as is in this article arrive without any mistakes when I send it by e-mail from Australia to Brunei? Full answers to questions like these must take in many factors. Here we will deal with only one: the use of mathematical algorithms to correct the errors that arise.

All of the above situations require information to be transmitted from one place to another, generally through many intermediate stages. A long distance telephone call, for example, begins with a person speaking into a handset. The equipment in the handset "samples" the person's voice. For long distance telephone calls, it might be sampled 8000 times per second and each of these measurements is converted to a number in the range 0 to 255. These numbers are then transmitted through a series of receivers and transmitters, at times over long distances, at times just from one component to another in a device and perhaps to and from a satellite. Eventually the information reaches the handset of the listener when the numbers are converted back to real sound. At every stage along the way, there are opportunities for errors to creep in. There will be errors in the measuring equipment, including errors occurring as the messages pass along wires due to interference from other electrical equipment, errors due to lightning and dirt and radiation. Receiving a good signal depends on two things:

- finding the errors and
- fixing the errors.

Mathematics is the key to both of these. Mathematical systems that can locate errors are called *error detecting* codes and I will briefly describe some examples of these, which are now familiar to most shoppers. Mathematical systems that can fix the errors are more complicated, so we shall look at these only briefly. They are called *error correcting* codes.

Examples of simple error detecting codes

The first example of an error detecting code is the supermarket barcode. This is a new application of old mathematics; number theory that is now several hundred years old. Supermarket barcodes are a series of black and white stripes representing the numbers 0 to 9 that commonly appear on items sold in shops where the prices are read by a machine. Items generally have 13 digits per barcode. Twelve of these digits provide information (such as the country, the manufacturer, the size, the product etc) and the last one is a check digit. The machine reader reads the digits and then calculates using an algorithm built into the machine whether the check digit is correct. If it is, the machine assumes there are no errors and processes the item. If there is an error, the machine is likely to make a noise requiring the operator to re-enter the item.

A supermarket barcode is based on the mathematical theory of congruence modulo ten. We will use the barcode 9 300650 654013 for Vegemite, a popular Australian food as an example. First every second digit is added: 9 + 0 + 6 + 0 + 5 + 0 = 20. Then the other digits are added and this total is multiplied by 3: 3 + 0 + 5 + 6 + 4 + 1 = 19 and 19 x 3 = 57. The check digit is chosen to make the sum of these two numbers a multiple of ten. The barcode for Vegemite passes this test because 20 + 57 + 3 is 80, a multiple of ten. Had there been an error in any one of the digits, the sum would not have been a multiple of ten and the test would have detected the error. Had the barcode been entered by hand, a transposition of two adjacent digits would have been a likely error. Most transpositions (but not all) would have been detected by testing the check digit. However, when an error is identified we do not know what it is. If the sum had been 81 instead of 80, any of the digits in the first total could have been out by one, any of the digits in the second sum could have been out by 7 (readers can check why this is) or the check digit itself may have been read incorrectly. Alternatively, there could have been more than one error. This is a simple error detecting code, which suits its purpose well. It identifies many errors. It cannot correct errors, but it is used in a situation where correction is not important – it is simple enough to request the information again.

ISBN Numbers

Another common example of a simple error detecting code is the ISBN book numbering system. There are ten digits in total, some representing country, publisher language etc and one check digit. It is selected so that the sum of 1 x first + 2 x second + ….. 10 x last is also multiple of eleven. Because eleven is a prime number, this system has stronger mathematical properties. The code can detect all one digit errors and two digit transpositions and many other errors. An example is 0 201 10238 2, the ISBN number of Thinking Mathematically (Mason, Burton and Stacey, 1982). The check digit 2 (the final digit) has been chosen so that 0x1 + 2x2 + 0x3 + 1x4 + 1x5 + 0x6 + 2x7 + 3x8 + 8x9 + 2x10 is a multiple of 11. If any one digit is wrong, the check digit cannot be right. If any two digits are transposed, the check digit again is not right. This property depends on the fact that eleven is a prime number. Again, this code cannot fix errors, but relies on repeat of the information when an error is detected. If the sum is one too large this might be because of a single digit error (such as the first digit is out by one, the second digit is out by 6, the third digit is out by 4 etc) or there may be an error in more than one digit.

Error correcting and error detecting codes

The ISBN code can be changed from an error detecting code to an error correcting code by adding a second check digit. If there are ten information digits, make the first check digit the number which when added makes the sum of all the digits equal to a multiple of eleven. For example if the ten information digits are 1234567890, the first check digit needs to be X (ten). This is because 1+2+3+4+5+6+7+8+9+0 = 45 and ten makes 55, a multiple of eleven. Make the second check digit the number which added to the multiplied sum makes a multiple of 11. In this example it would be 1 because 1x1+2x2+3x3+ …. + 9x9+0x0 = 285 and 286 = 26 x 11. The full code word is therefore 1234567890X1. It can be proved that this system identifies and corrects all single errors and detects all double errors. Readers with an understanding of congruence and multiplicative inverses modulo eleven will be able to prove these results.

Today's codes

It is not possible to make a code that can correct any error. Even transmitting all the information two or three times will not be fool-proof. The secret to constructing good codes is fitness for purpose. Error correcting codes are devised so that they can deal with the type of errors that are likely to occur in the situations where they are applied. In 1971, a 32 digit code with 26 check digits was used to transmit the first pictures from Mars. This code was capable of correcting up to 7 errors in each piece of information (each "codeword"). Signals were received on earth from a 20 watt transmitter, less than a domestic light bulb. Because the signal was extremely valuable, it is worth having nearly half the signal in "check digits" rather than giving new information. In contrast, long distance telephone calls use a 255 digit code with 24 check digits. They can correct up to 3 errors. In this circumstance, the signal is not as valuable and so having fewer check digits is simpler but still satisfactory.

Compact discs reached the commercial market just a few years after the error correcting code on which they are based was invented at Phillips in 1970. The high quality of compact disc sound relies on the on-board computer detecting errors (in reading the disc as well as in recording it) and correcting them before play. To play music from a compact disc about 1.5 million numbers are read per second. Only about a third of these are the actual music. The code for compact discs is designed to cope with bursts of errors.

The basis of modern codes is number theory, geometry, statistics and other branches of mathematics. New codes are constantly needed for new applications. Many of the new codes are based on highly abstract mathematics, which appears at first to be far removed from the real world. Readable accounts of these are given by Barnett (1995) and Holton (1995). One of the approaches begins by regarding the strings of numbers that are the codewords (e.g. 9 300650 654013 for Vegemite) as a point in space. The Cartesian (x, y) plane studied at school takes a string of two numbers and interprets it as a point. For example, the string 43 can be interpreted as the point (4,3) in two-dimensional space. Analogously, the codeword for Vegemite can be thought of as a point in 13 dimensional space. Of course, 13 dimensional space is a very abstract concept – our real world is only 3 dimensional, but extending the idea of dimensions in this way turns out to be very powerful. In the address, I demonstrated intuitively that good error-correcting codes can be made by using as codewords, the centres of "spheres" which are packed as closely together as possible. For a good code these spheres are multi-dimensional. The codewords are points in a multi-dimensional space and they are separated by multi-dimensional "spheres" of words which can be recognised as containing errors. When an erroneous codeword is received, the code replaces it by the codeword at the centre of the sphere which contains it.

There are many codes based on packing of multi-dimensional spheres. For example, mathematicians created a code for mobile radio by studying the packing of 100 dimensional spheres in 100 dimensional space (Conway and Sloane, 1988). In two-dimensions, the way in which the cells of a beehive are packed is hexagonal close packing. In three dimensions, the way a green grocer stacks oranges or other spherical fruit is a close packing. The generalisation of these solutions to natural real world problems to many dimensional space is the secret to create some of the best codes that we know.

Lessons for mathematics education

The mathematics behind some of the new appliances of the digital era is, I hope, interesting. However, for this address, I want to use it as a stimulus to discuss how mathematics education relates to national development. First, I think it is clear that mathematics is useful and there are new uses not dreamed off just a few years ago. School mathematics sometimes gives the impression that discovery was finished many years ago and is now simply being implemented. This is quite wrong.

Secondly, some useful mathematics is very abstract. In some of our curriculum thinking, there is a tendency to think of the only the simplest mathematics as useful and abstract mathematics as an indulgence.

Thirdly, mathematics is being embedded in objects. If we look around in a supermarket, a few customers and a few salespersons may be doing some calculating, or estimating or measuring something or deciding whether a particular shaped box will fit into a corner – all of which use mathematical thinking. At the same time, all the cash registers, all the scanners, the inventory systems, the electronic banking system and even the mobile phones in the customers' pockets are carrying out millions of calculations. Consequentially, the mathematical needs of citizens are changing, for everyday life, for commerce and business and for high technology and science.

The curriculum content challenge

The digital revolution presents us with what I see as an unprecedented need to rethink the mathematics curriculum. Routine calculation is becoming the work of machines, rather than people. Basic arithmetic for shopping, statistics for a community group, graphing in school mathematics and algebra for an engineer - all are becoming affordably automated. A machine capable of doing school and university algebra cost some thousands of dollars in 1980, it costs some hundreds of dollars in Australia now but predictions are that in 15 years time, it is likely that it will virtually disposable and of negligible cost.

At the same time, abstract concepts are proliferating. In the coding theory example above, we briefly touched on one of the many ways in which this is happening. We made a correspondence between strings of numbers and points in space – and it was not ordinary space, but multi-dimensional space.

This poses enormous challenges for mathematics in schools, especially in deciding what to teach and to whom. The content that has been the backbone of the school curriculum in the twentieth century is the teaching of routine algorithms. It may not have been done as well as we may like, but routine procedures are possibly the easiest part of mathematics to teach. If the success of education systems worldwide is taken as evidence, then it seems that this is rather easier to teach than sensibly using algorithms, estimating answers, deeply understanding concepts and seeing how they can be applied. These goals have long been recognised as the most important goals of schooling. In the new technology environment, pursuing the other goals will be of little value. The presence of four function calculators (most likely embedded in objects) will eventually lead to a thorough reassessment of arithmetic and the new algebra technology will lead to a re-assessment of most of the standard fare of secondary school mathematics.

To give just one example, our recent research project "Learning Algebra" at the University of Melbourne has attempted to uncover some of the ways in which algebra teaching will need to change (Stacey and MacGregor, 1997). We have looked at two aspects of algebra that we believe will remain fundamental. Reading and writing the language of algebra precisely seems essential, as a way of communicating with machines, so that problem situations can be described and solutions can be interpreted in human terms (MacGregor & Stacey, 1997). However, formulating problems algebraically (usually as equations) presents cognitive challenges far beyond the language aspects. For example, identifying the variables involved and noticing functional behaviour and necessary relationships are difficult steps requiring a new "algebraic" way of thinking, not just an extension of arithmetic thinking into a domain of letters (Stacey and MacGregor, 1997a).

It is pleasing to see work here in Brunei Darussalam and reported at this conference is joining the worldwide effort to rethink what mathematics should be in the future.

The perennial problem of mathematics teaching

The coding theory example provides an excellent example of abstract mathematical ideas, which probably seem intensely abstruse to a novice, which are paradoxically useful. Yet all mathematical ideas are abstract. Even when a child first learns subtraction, the subtraction of the numbers is abstract, not concrete. We may put 5 objects in front of a child and take away 3 of them, but the subtraction lies in the taking away, not in the objects that can be seen or felt.

Perhaps surprisingly, successful mathematics learners do not feel that mathematical objects are abstract. Instead they feel concrete. The coding theory example illustrated one way in which this happens. Strings of numbers begin to be identified with points in space. Instead of thinking about the string of numbers, the codeword, we draw on spatial analogies and think about points and spheres. And although the dimension of the space is "unrealistic", we use concepts such as packing from familiar two and three-dimensional space to make the multi-dimensional space feel familiar. When concepts feel concrete in this way, we can then use them to build other concepts and so the spiral of abstraction continues. (Mason, Burton and Stacey, 1982) Making abstract entities feel concrete is the perennial problem of mathematics education.

The feeling that these abstract mathematical concepts are somehow concrete is, I contend, a sign of understanding. Teaching rules without reason cannot create it. One example of this that we have encountered in our research study on the learning and teaching of decimals is with rounding numbers. Too often, our students are taught to round numbers just by rule: if the digit following the one you want is less than 5, just knock it off; if it is 5 or more then increase the last digit you want by one. Disappointingly, some of our textbooks explain the ideas no better than this. Evidence that this teaching has not worked is easy to find in the newspapers, where ridiculous figures are very frequently given. Often we read police reports of a robber, only dimly glimpsed, who is 183 cm tall. Our finest newspaper recently had a centre-page spread on a mountain range that is 6027 metres high. It certainly seems that the writer had not appreciated why rounding numbers to a certain number of significant figures is sensible. A reader of the New Scientist recently quoted a recipe for soup that uses one large onion and 397 gms of tomato puree. The accuracy of one ingredient contrasts bizarrely with the inaccuracy of the other. It is particularly bizarre because the tin on which the recipe is written contains 400 gms!

It seems likely to us that one of the reasons why such as simple rule as the rounding rule is so badly remembered and why significant figures are so much abused is that children's understanding of decimal numbers is quite inadequate. Our estimate (Steinle and Stacey, 1998; Stacey & Steinle, 1998) is that about 25 per cent of school leavers have serious difficulty here and so it is likely that a good proportion of the adult population do not understand decimal notation, and consequently cannot make real sense of the rounding rules. In the address, I will show an interview with a child who has a consistent but wrong idea of decimal number, which affects her work in many areas of mathematics. Research into children's understanding is the key to teachers' knowing how children are likely to interpret what they say. Children who have, for various reasons, built an "alternative framework" for a mathematical concept are not going to have a secure foundation to which new knowledge and skills can be added. Instead, rules will be added onto a loose pile of unconnected ideas and the whole conceptual structure will feel anything but concrete.

Conclusion

In the address, I have tried to provide an example of mathematics that is underpinning the digital revolution. Knowing and understanding mathematics of this nature will not be for most people in a society, although for national development it is important that a country has some people able to operate on this level. The technological revolution puts mathematics education in a markedly changed circumstance. There is less, if any, need for low level mathematics, for the routine procedures that can be automated. The need for strong sensible understanding of mathematical ideas at the everyday level (estimating, recognizing links, testing hypotheses quickly etc) has possibly stayed the same. In most countries that will participate in the digital revolution, the need for mathematics at a high level will probably increase. We have discussed some aspects of the unprecedented curriculum challenge that this provides for mathematics educators and school leaders. At the same time, research into children's thinking is making some progress on the age-old problem of mathematics teaching, to make abstract ideas feel concrete.

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