Mathematics - what should we tell the children?

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Today, new technologies are always in the news, especially the internet. In the public domain, one of the principal debates about internet use has been the question of censorship. The internet can give children free access to "adult" and other material which most of society believes should be restricted. In response, software such as NetNanny has been enthusiastically greeted by some schools and some parents to provide a form of censorship on the internet, protecting children from material that is not suitable for them.

My question in this paper is to explore how we might react to calculator and computer systems that will potentially expose students to material for which they may not be "ready". Do we need "mathematical netnannies" for calculators and computer algebra systems, to protect children from the advanced mathematical ideas that are built into these systems? It would be easy enough to buy them, if there was teacher demand. What is the best way for teachers and children to deal with the deep secrets of mathematics that are embedded in calculators and mathematical software but which conventional curriculum thinking regards as beyond them? Should we protect children or openly display these secrets? Alternatively can we use the allure of the mysterious to entice students to explore mathematics more widely than has been possible in the past?

Thinking about these questions brings us to central questions about the mathematics curriculum and about learning and about mathematical practices. In the rest of the article, I will illustrate some of the issues involved with a number of examples. Because there has never yet been widespread availability of mathematical systems which really challenge the secondary curriculum, we will first look at the situation in primary schools where the four-function calculator has posed deep challenges. Scientific calculators were able to be incorporated into the secondary curriculum with little change to the main thrust of the subject. In contrast, widespread availability of computer algebra makes us reconsider the role of the central concepts and skills of secondary mathematics, just as there has had to be a reconsideration of what arithmetic, the central thread of the early mathematics curriculum, should be about in primary schools.

Four function calculators in infant grades

In the early years of primary school, children deal exclusively with small counting numbers: 1, 2, 3, 4, 5, 6, ... generally reaching three digit numbers within three years. Introducing a simple calculator into this curriculum regime changes the scene enormously. Such an situation was documented by the *Calculators in Primary Mathematics* project (Stacey and Groves, 1994). Four function calculators were provided to all children on school entry at six Melbourne schools over the years 1989 -

1993 and the cohorts of children were followed as they progressed, with their calculators, through the grades. The children were tested on a variety of aspects of mathematics and their teachers were regularly interviewed.

The presence of the calculators gave access to three sorts of numbers that children of their age had rarely encountered in the past: they encountered much larger numbers than before, they found negative numbers which had previously not been acknowledged in primary mathematics and they came across decimals much earlier than was planned. How did the children and the teachers react?

Large numbers

Children have probably always been fascinated by large numbers, but the usual classroom mathematical tools (written symbols and concrete materials) do not facilitate exploration. In contrast, calculators enabled many children to play with large numbers and see the patterns inherent in our numeration system unfold before their eyes. The calculators had a "constant addition" facility, which enabled easy counting and skip counting (e.g. pressing the buttons 3, +, 2, =, =, =, ... causes the calculator to display in turn the numbers 3,(2), 5, 7, 9,...). Children observed, for example, when counting by 9's the units digit decreases by one most of the time, while the tens digit increases by one. Children could watch the units cycle around and the tens, hundreds and even thousands clock over as they used their calculators to count up and up to previously unscaled heights. Some teachers welcomed this and encouraged the free exploration and sharing that enabled it to happen - others had concerns.

Some of these concerns were for the children who could not participate: many teachers reported that the differences between their students understanding was apparently growing. Whether this was because the differences did stretch out or because the differences became easier to see is unclear. One teacher of Grade 3 commented:

> *I think [using calculators] helps you deal with the different levels in the grade. It's helped me to understand their thinking processes probably because it frees them up and there is no threat about doing things the correct way and getting the correct answer. Everybody understands that that's accomplished already, so it's what you think about it and what you are going to do about it, that's important. Once we establish that - I think I'd have been really lost with some children if I hadn't had it.*

A second cause of concern was whether the children who were confidently talking about numbers such as fifty thousand and trying to count up and up with their calculators "really understood" what such a number means. Some children gave clear evidence of a degree of understanding. In one prep grade, Ben had counted up to 17 900 by 100's. When asked what number would be reached after pressing equals two more times, he wrote 18 100, although he read it as eighteen hundred and one. Other children may have been copying down numbers from the calculators without thinking what might come next.

There is a strong and well-based principle that the best way to learn mathematics is to "really understand" the entities that you are working with at every stage. Simply following steps, without having a strong meaning at every stage, leads students to memorising mathematics as a set of procedures rather than understanding how and why it works. However, too much emphasis on "understanding" before doing can lead to a constrained and boring curriculum, restricting the range of ideas that students are exposed to. Some educators believe, for example, that children in the first year of school should only work with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and possibly 10 because they cannot fully understand larger numbers. In contrast, I believe that understanding does not come in just two states (you either have it or you don't) but is something that grows. I am happy to see children playing with large numbers, provided that some of the many things there are to come to understand are being worked on. A child who counts for pleasure to 850 by tens (using the constant facility 10 + = = = = eighty five times) and then counts back to zero by ones (using the constant facility 850 -1= = = eight hundred and fifty times) will be learning something about the size of 850, something about the ratio of 10:1 and about the patterns in base ten numeration. This, I think, is an example of how understanding about number grows. It should not be forbidden or discouraged because children "don't understand" 850. Groves (1995) discusses the tensions felt by one teacher between the usual curriculum goals and the developing understanding of number shown by her students.

Negative numbers

Negative numbers were generally first found by accident. Many children, counting backwards with the constant addition facility, accidentally went too far. In some classes, children had surprisingly little difficulty giving a meaning to negative numbers. In Figure 1, Kylie's picture of numbers is shown. Her prep class had been discussing hibernation and they had named these the underground numbers. Small children have lots of questions about negative numbers (is there a "minus nought" for example), but number line models such as above/underground numbers provide a strong intuitive picture which they can use to answer questions such as "what is 3 more than -2?". (Note that this does not mean they would make any sense of the calculation -2 + 3.)

Teachers' attitudes to introducing negative numbers varied considerably. At one extreme, Kylie's teacher encouraged her to talk about and to spend time illustrating her ideas of negative numbers. Teachers at the other extreme avoided them. One Grade 3 teacher between these extremes described her approach thus:

> *Sometimes when we are playing a game they go too far backwards and they end up with a negative answer and they say "what have I got'" But I haven't really pushed it, I've just said that it was a negative number, a number lower than zero. We have zero on one line and all the ones above zero (indicates with hand) and all the ones below zero (indicates with hand). Those children who can understand initially, I tell them but then there are some who are completely lost so I just let them go. We haven't touched negative numbers as such in Grade 3 but through the calculator they've come up.*

Children in primary school have in the past been shielded from negative numbers. The standard teacher's answer to two take away five has been "cant be done". With a standard four function calculator, children can find out almost for themselves, that there are mathematical secrets behind the teacher's glib denial. It would be possible to modify a calculator that answered "cant be done" instead of offering negative answers, but would children be better off with it? My feeling is no.

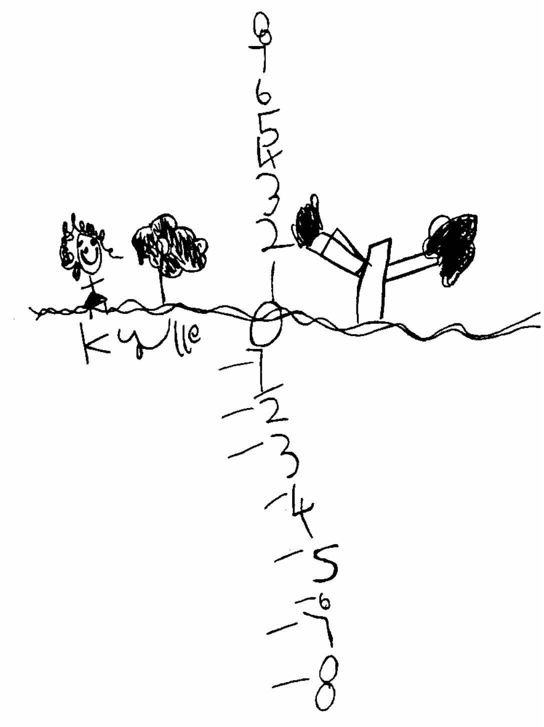

Figure 1. Kylie’s underground numbers

Decimal numbers

Just as subtraction of counting numbers leads outside the counting number system, so does division. In classes where children used their calculators to help in real problem solving, decimals were quickly encountered. This is in contrast to what might be regarded as the normal situation, where teachers carefully contrive the division questions so that everything "goes" until quite late in primary schooling. One instance where we observed children coming up against decimals occurred on Teddy bears' picnic day when Zoe and Julienne tried to share 55 teddy bear biscuits fairly amongst 10 teddy bears. Several days later, they explained to our project visitor that the division number sentence showed that each teddy bear got 5 and a bit biscuits each. They were not able to interpret the decimal part in a precise way, but they knew that 5.5 was more than 5 and less than 6. Their work is shown in Figure 2.

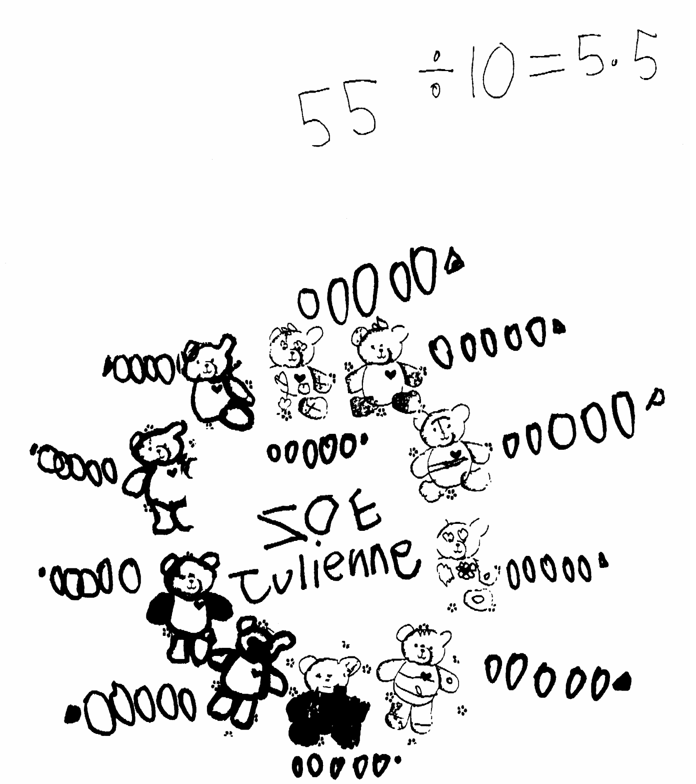

Figure 2. Zoe and Julienne give each teddy 5 and a bit biscuits.

Another teacher of Grade 3 recalls an early encounter with decimals this way:

> *When we were making damper we were estimating how much water it would take to make the flour into dough. We had put in one cup, then someone said we'd better add half a cup now, and they wrote the half fraction on the board. When they were adding it up, some kid just got the calculator, and I said to add it up, and didn't say anything else. He said "How do I add half?" and I said, "Well half on your calculator is .5; you can write half like that or .5. Really it was just incidental teaching. Now to those who are ready to accept it, it just became part of them right then and there because it was understood. I noticed it not too long ago when we did Aborigines last term and I remember it came out so matter of fact. The children were saying 'half' and on the calculator it was .5. Yes, that's about it, we went no further than that.*

How widespread is knowledge of the new numbers?

As one part of the formal testing program, we interviewed a 20% random sample of the Prep. Grade 1 and Grade 2 children at two of the project schools. Children were shown "large numbers" on flashcards and asked "Can you tell me what this number is?". The percentages of children who could correctly read large numbers (and a decimal for grade 2) from a flash card are shown in Table 1. A significant proportion of the children were able to read numbers which are usually considered well beyond the curriculum at each of the grade levels, as specified by the Victorian *Curriculum and standards* *framework: Mathematics* (Board of Studies, 1995). Such numbers are indicated by shaded cells in the table.

Table 1

Percentages of children able to correctly read given numbers at different grade levels

Level		Numbers and % correct responses*			
Prep (n = 29)	Number	74	126	203	
	% correct	46	7	4	
Grade 1 (n = 24)	Number	203	1435	62 750	1 000 000
	% correct	58	46	21	12
Grade 2 (n = 23)	Number	87	372	3294	5.7
	% correct	100	100	65	39

* shaded table cells indicate numbers beyond the normal curriculum

In addition, 30% of grade 2 children (the only grade level asked) were able to correctly explain the meaning of –5 as "5 below zero" or similar. Only 9% of grade 2 children when asked "how big is 5·7?" were able to give a correct answer such as "a bit bigger than 5". Thus, many children showed some understanding of negative numbers, many could also correctly read a decimal, but relatively few could give an indication of its size — for example by comparing it with a whole number.

In another part of the testing program (Stacey, 1994) we compared the ability of Grade 3 and 4 children who had been in the project for at least 3 years (346 children) and a control group of 593 children to use a calculator to do questions in arithmetic such as 186+492, 1000000-192, 187 + 4.92 and 86 x 21. Both groups handled whole number calculations equally well, but the project children were markedly better able to interpret answers involving decimals or negative numbers. Children not in the calculator project frequently did not see the decimal point or the negative sign and wrote down nonsense answers such as 21 ÷ 84 = 025 or 25. Children with the calculator experience may not have completely understood the answers they wrote but many fewer wrote down nonsensical answers.

Computer algebra systems

Complex numbers

If we allow very young children to use four function calculators, the *Calculators in Primary Mathematics* project found that they will very likely be taken beyond the bounds of the normal curriculum by indulging their own interests (large numbers), by accident (negative numbers) and by trying to solve real problems (decimal numbers). We found that some teachers saw this as a good thing and encouraged the exploration and sharing that promoted it, whilst others tried to organise their classrooms so that these new types of numbers remained a secret. Some children required very little support to assist them to make some intuitive sense of the new numbers that the calculators presented them with, whilst others made no attempts at all to engage.

What is likely to happen with a computer algebra system? Solving polynomial equations, even with easy coefficients, will lead to complex numbers in the answers. The decimal numbers and the negative numbers encountered by the primary children both have interpretations which seem to a modern person to be fairly straight forward - decimals fill in between whole numbers and negatives extend the number line in the opposite direction. Historically it took many centuries of unease before people accepted negatives as numbers. One major step was in the 14th century when they were seen to unify the various methods for solving cubic equations without causing wrong solutions to arise (Boyer, 1968). From our perspective there do not seem to be great difficulties for students to get an elementary grasp of the ideas. But what of complex numbers? Could we give students an elementary and intuitive idea about them? What would it be and what would it help them to do?

Advanced functions

A computer algebra system can solve equations with algebraic letters as the coefficients. Even if the solutions are not complex, the mathematics needed to understand a solution can be well beyond the mathematics needed to pose the question. This is a very common situation in mathematics: very large new mathematical structures have often been created in an attempt to solve problems that are posed within the old mathematical structures.

As an example of this process, let us look at the solving of cubic equations. The work that revolved around solving arbitrary cubic equations in Italy in the 14th century was critical in the history of algebra. It resulted in a much better understanding of how algebraic letters could be used, in the acceptance of negative numbers - finally- and in advances in the understanding of complex numbers. For an outline of the history of solving cubic equations and a more complete discussion of the problem below, see MacGregor, Stacey, Pegg and Redden (1994).

If students had a computer algebra system, it would seem obvious that they would be encouraged, rather than forbidden, to solve cubic equations, such as those arising from volume problems. Solving the cubic equation

$$x\,(L - 2x)^2 = 0.05$$

will find the depth (x) of an open box of volume 0.05 m^3 made by folding up a square piece of tin of side L metres with the corners cut off. (See Figure 3)

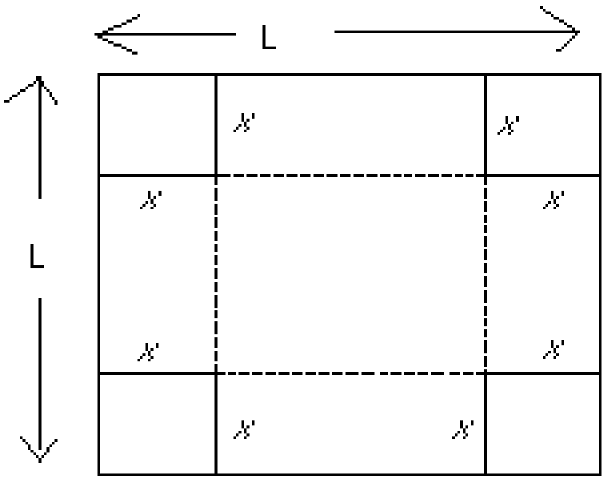

Figure 3. Pattern for folding a box from a square piece of tin.

If L= 1, using DERIVE gives three real answers: x = 0.066, x = 0.294 and x = 0.639. The first answer gives a shallow tray, the second nearly a cube and the third, although it is a real number, does not give a physically possible solution. This is useful and within the capacity of students with a little knowledge of volumes and of algebra. However, if we use an unknown L instead of L = 1, DERIVE can again be used but it gives the solutions:

$$x = \frac{L}{3} + \frac{L\cos\left(A - \frac{5\pi}{6}\right)}{3},$$

$$x = \frac{L}{3} + \frac{L\cos\left(A - \frac{\pi}{6}\right)}{3},$$

$$x = \frac{L}{3} + \frac{L\cos\left(A + \frac{\pi}{2}\right)}{3}$$

where

$$A = \frac{\text{artan}\left(\frac{\sqrt{3}(20L^3-27)}{9\sqrt{40L^3-27}}\right)}{3}$$

It is much harder to understand this solution than it is to understand the original question. A student would need to know about cosines of angles in radians and about the artan function; knowledge which in the past has not been needed in polynomial equation solving at school. On scientific calculators, there are buttons for cosine, for artan and for working in radians but somehow, these buttons have not impinged on other parts of the curriculum. Some students may have experimented with them briefly, but generally their presence has caused no threat to the order of the curriculum. The trig functions could be kept in trigonometry until we are given an advanced tool like computer algebra which uses mathematics at the advanced level where all these ideas are tied together.

Exploration in the secondary school

There is great opportunity for exploration with a computer algebra system. For example, a student who was encouraged to experiment with solving polynomial equations and factorising would probably soon observe some of the important properties that are generally beyond common discussion at school - the fundamental theorem of algebra (that a polynomial of degree n has at most n roots), the idea of repeated roots, the fact that the complex solutions of a polynomial with real coefficients come in complex pairs, the fact that polynomials of odd degree always have at least one real root which is of course also easily deduced from a knowledge of the general shapes of the graphs.

We noted above the way in which some of the infants' teachers encouraged children in their class to use their calculators to explore new mathematical ideas and the ways in which they were prepared to contemplate changes to their curriculum. Perhaps sadly, it has not become commonplace for teachers to report secondary students being excited by what they find on their scientific calculators - despite the potential that is already there - calculators convert numbers to other bases, for example. Even one of our Grade 3 teachers noted:

> *At first I found it a bit hard with the Year 3 children, because the Preps are so exciting. Everything is so exciting to them and the whole deal of using the calculator is very exciting. Well, it's not the same, it much more mundane by the time they're to Year 3.*

By and large, however, scientific calculators properly used do not throw up surprises: the type of output is closely related to the type of input (perhaps scientific or engineering notation for very large or very small numbers being the major exception) Even if a calculator has a complex number facility, it will not give complex answers unless it is deliberately put into complex mode.

Knowing what you are doing

Four function calculators do not help in problem solving unless a child can identify which operations (+, -, x, /) they need in which combinations to solve a problem. This has always been one of the most difficult parts of arithmetic to teach and the calculator cannot of itself help. A child must recognise, for example, that to find the cost of 0.56 kg of meat costing $8.95 per kg they must use multiplication, not division even though they want an answer less than $8.95. Interestingly there is some evidence (Hembree and Dessart, 1986; Stacey and Groves, 1994) that instruction with a calculator may be able to assist children to learn to identify operations. The reasons for this are hypothesised by Stacey and Groves (1994).

With any other mathematical system, be it algebra, statistics or anything else, the problem is also one of identifying what to do. The video by Tynan (1996) illustrates this nicely with an example of fitting regression lines to data. The data is of the temperature of water cooling from 100^{o}C at 30 second intervals of 5 minutes and we want to predict the temperature later. The regression menu provides a choice of linear, polynomial, exponential and logarithmic regression. Knowing something about cooling, we choose exponential regression and immediately a curve appears, seductively passing near all the points (se Figure 4a) However, this exponential regression is not a reasonable mathematical model of the situation, despite the fact that it apparently fits the points well. Our menu choice in fact fitted a curve of the form $y = Ae^{-bx}$ and such a model must asymptote to zero, thereby predicting that the water would eventually freeze. To use the automatic exponential regression meaningfully for a valid prediction, we had to carry out the regression not on the temperatures but on the differences between the temperatures and the ambient temperature as in Figure 4b. This example illustrates that a lot of knowledge is needed to use technology sensibly - teachers certainly wont be out of a job.

Figure 4a and 4b: stills out of the video with titles as follows
Figure 4a. Exponential regression of temperatures against time
Figure 4b. Exponential regression of temperature difference against time

Analogies between primary school arithmetic and secondary school algebra

In the coming decades we have a number of choices to make about how technology and the mathematics curriculum interact. Our experiences with introducing four function calculators into primary schools are to some extent analogous to the introduction of computer algebra systems into secondary schools.

Both challenge in a central way the thrust of the curriculum - the four function calculator does the algorithmic arithmetic that still dominates primary school mathematics and the computer algebra system does the algorithmic algebra and graphing that forms the central thread of secondary mathematics. In both cases, the routines that teachers spend a lot of time on (solving linear and quadratic equations; addition, subtraction, multiplication and division) are in theory replaceable. In the case of primary school arithmetic, we have come along way (but have not reached the end of the journey) in identifying what students need to be able to do by themselves and what it is sensible for most people to rely upon a calculator for. Nobody now thinks that students do not have to learn about the operations of arithmetic and nobody thinks that the old curriculum was appropriate. Ideas of number sense, estimation and mental calculation supporting calculator use are becoming widely accepted. In the case of computer algebra we have yet to reach any consensus of what "algebra sense" might be? Which of the following linear equations, for example, would we want students to be able to solve for x without use of technology

$$2x+ 3 = 15, \quad 3x + 4 = 7 - x, \quad ax + 2a = 7x \ ?$$

I would argue now that all of these linear equations are analogous to simple calculations like 35 + 60 which I would want all students in primary school to be able to do quickly without resource to any aids, but we have little experience on which to base these judgements.

Both technologies need substantial skill to be able to be used well. Chris Barling, in his conference address reporting on his experiences using Maple with tertiary students, commented "as soon as you do simple things, complicated things arise. " Four function calculators for infant grades and computer algebra for secondary schools are both tools substantially more advanced that the mathematics being dealt with at that stage. This means that there are nasty surprises in store or great learning opportunities, depending on what students and teachers make of them.

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