BUILDING FOUNDATIONS FOR ALGEBRA

Learning algebra is an important milestone in students' mathematical development. It opens the door to organised abstract thinking and provides a tool for logical reasoning. It gives students the satisfaction of finding simplicity in what appeared to be complex and finding generality in a collection of particulars. For example, phenomena as diverse as the growth of algae in a pond and accumulating bank debt manifest the same mathematical behaviour; in each case a small rate of change accumulates over time to produce substantial changes in the totals, and even a tiny change in percentage growth can produce unexpected and devastating effects. This behaviour is captured algebraically in the exponential function, but it is hard to describe clearly in words as our attempt above shows. For this reason, the language of algebra is the standard medium for precise communication about numbers and functions and is essential for higher mathematics. It is important that students learn to master this new language.

Over several years of research, we have gathered empirical data about the types and causes of specific difficulties encountered by students beginning algebra. Many of their difficulties can be traced back to limited understanding of number and operations. Other errors are caused by not knowing how to write what they do understand.

Our work shows that the best preparation for learning algebra is a good understanding of how numbers work. Developing and extending this understanding is an important job for teachers throughout the middle grades.

In this article we describe some basic difficulties in early algebra and their causes, and we present some practical strategies for overcoming them. We deal with the following five aspects of arithmetic that are essential foundations for learning algebra:

- seeing the operation, not just the answer.
- understanding the equals sign.
- knowing important properties of numbers.
- being able to use all numbers, not just whole numbers.
- working without a practical context.

We suggest some activities that are already in mathematics programs for the middle grades, which could assist students' algebra if they were used more frequently or given a different emphasis. These activities provide a focus for number work at various levels, and can be incorporated into existing programs without needing extra time. Teachers will find more ideas for exploration of number properties and processes in the *Curriculum and Evaluation Standards* (NCTM, 1989).

SEEING THE OPERATION, NOT JUST THE ANSWER

The problem

In a recent study we gave 14-year-old students tables of values such as that shown in Figure 1 and asked them to say what they thought the value of y would be for some larger values of x. Many gave correct answers (e.g., 804 for the Figure 1 table when $x = 800$). However only three quarters of the students who gave the correct numerical answers were able to correctly describe in words the relationship between x and y that they had just used (e.g., "add 4 to x") and only a half were able to write it in algebraic symbols (e.g., as $y = x + 4$). How can this be? These students had just discovered the relationship for themselves and used it correctly, yet so many could not say in words or in symbols what they had done.

x	1	2	3	4	5	6	7	8	9	10	..
y	5	6	7	8	9	10	11	22	13	14	..

Figure 1: Table of related values of *x* and *y*

When we interviewed students we found that they often saw patterns in the numbers that could not be expressed algebraically. Some of them said, correctly, that as x increases by 1, y increases by 1. Others talked about the gap between x and y, for example, "From x you miss three numbers then there is y", or they noticed that each number appears four places earlier in the bottom row than in the top row. These ideas are sufficient for calculation but they cannot be written in algebraic notation. To use algebra, students have to be able to see and say the addition or subtraction operation in any of these equivalent ways: $y = x + 4$, $x + 4 = y$, $y - 4 = x$, $x = y - 4$, or $y - x = 4$

There are often a lot of ways for working out a problem in arithmetic. For example, Figure 2 shows five solutions for one question. Although it is primarily a division question, it can be done by addition, subtraction, multiplication or division. However, in algebra, if the target is T items and the class bought D items per day, there is only one way of writing the number of weeks it will take: $\frac{T}{5D}$. With numbers, students can do this problem without seeing it as a division question. With algebra, there is no option.

Some Suggestions

- Observe and discuss in class the variety of methods that students use to solve problems such as SCHOOL FETE (Fig. 2), but try to move them on to the more sophisticated methods (e.g., using division or multiplication rather than trial additions and subtractions). Thornton (1985) provides an interesting discussion of what procedures students actually use to solve problems. He proposes that people calculate in everyday life as if there are ten operations, not four. For example, they often treat doubling and halving as operations distinct from multiplication. For algebra, the "ten operations" must be reduced to four.

SCHOOL FETE
Our class is collecting items for the white elephant stall at the school fete. If we collect an average of ten items per day, how many weeks will it take us to collect 500 items?

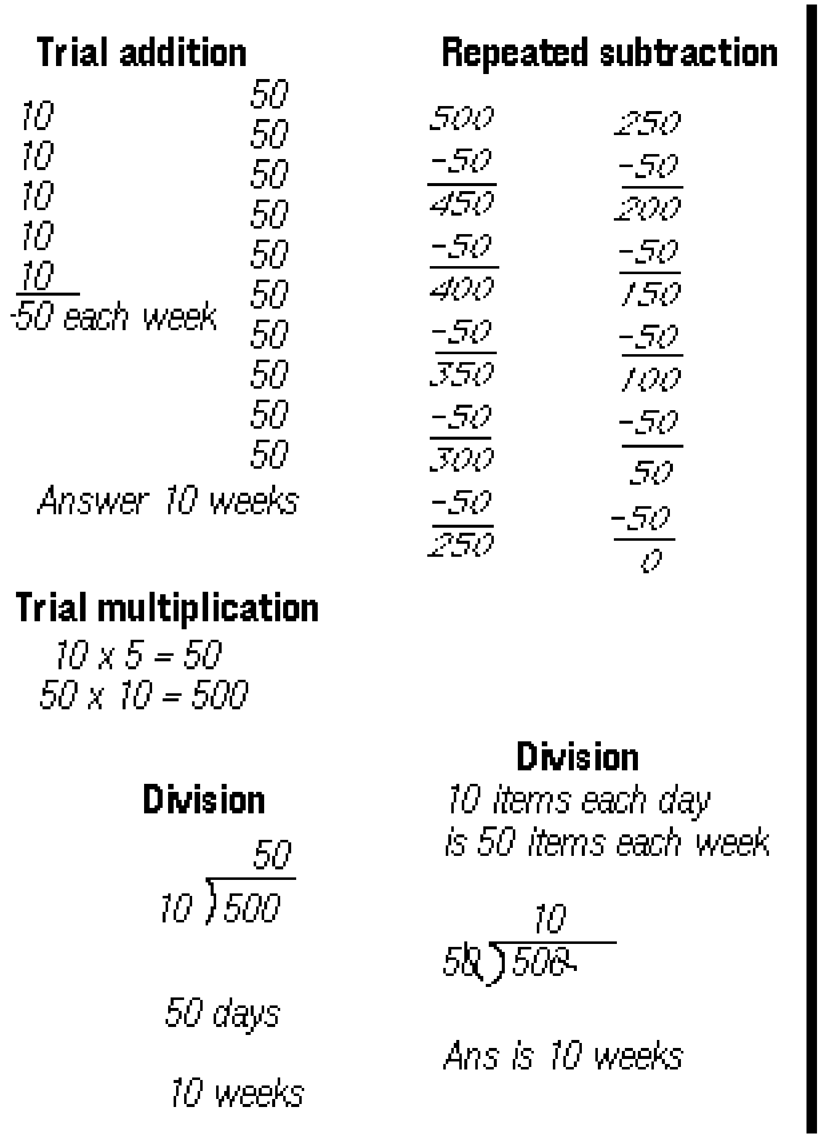


Figure 2: Different ways of solving a division problem

- Frequently allow students to work out questions with calculators, especially questions where the numbers are large. Often this helps to crystallise thinking about which operation to use. Fielker (1986), for instance, reports how several ten year olds, who were asked to enter 10 on their calculators and then halve it, simply subtracted 5. When they were then asked to enter 11 and halve it, they didn't know what to do. They could find a half of 10 mentally but did not see it as a division by 2.
- Check your textbook series and worksheets to see that the operations are presented in all the common ways in which they occur in real problem settings. For example, some worksheets only present subtraction as "take away", but subtraction occurs in real world problems from situations about changing, equalising and comparing (Fuson, 1992). Some examples are shown in Figure 3. Students will not be able to reliably identify an operation as subtraction if they only associate it with classic "take away" problems.

(i) Change take from: missing end.

Matthew had 15 lollies and he ate 9. How many did he have left?

(ii) Change take from: missing change.

Nicholas had 15 budgerigars. Some flew away and he had only 9 left. How many flew away?

(iii) Compare: difference unknown.

Carol is 15 years old. Leila is 9 years old. How much older is Carol than Leila?

(iv) Compare: difference known

Carol is 15 years old. She is 9 years older than Timmy. How old is Timmy?

(v) Equalise: difference unknown

Tom has 15 key rings. Adam has 9 key rings. How many key rings would Tom have to lose to have as many as Adam has?

Figure 3: Some different types of subtraction problems

- Continue using number pattern work where students have to predict what the next numbers will be, but also encourage students to say precisely what the pattern is, in terms of arithmetic operations as well as in less formal language. The example in Figure 4 shows some of the variety of ways in which simple mathematical relationships can be expressed. Students need to hear all these forms used by their teachers, their classmates and themselves, and they need practice at changing one form to another.

- Most of the number pattern work in early grades is centered around the successor relationship: finding the next member of a sequence from previous members. So, when older students are presented with the sequence: 1, 4, 9, 16, 25, 36, ... they can continue it by noticing that the differences are increasing odd numbers:

  4 - 1 = 3, 9 - 4 = 5, 16 - 9 = 7, ...

  Being able to see these increasing differences is an important strength. However, it is appropriate to draw their attention to the functional relationship, i.e., the fact that the sequence is also

  1 ∞ 1, 2 ∞ 2, 3 ∞ 3, 4 ∞ 4, 5 ∞ 5, 6 ∞ 6, ...

  As another example, many students should recognise both successor and functional relationships in the sequence

  18, 27, 36, 45, 54, 63, 72, ...

  They will see that the tens digit is increasing by one and the units digit decreases by one and that the numbers are increasing by 9, but they also need to see that the numbers are 2 ∞ 9, 3 ∞ 9, 4 ∞ 9, 5 ∞ 9, ...

EXPRESSING RELATIONSHIPS

The ages of Niree and Jonathon as they grow up are shown in this table. Write down how their ages are related.

J's age	0	1	2	3	4	5	6	7	8	.
N's age	6	7	8	9	10	11	12	13	14	.

Possible answers:

Jonathon is 6 years younger than Niree.

There is a 6 year difference between Niree and Jonathon.

To find Jonathon's age, you take away 6 from Niree's age.

Jonathon's age is equal to Niree's age minus 6.

Niree's age is equal to Jonathon's age plus 6.

Niree is 6 years older than Jonathon.

Figure 4: Students need to practise different ways of expressing a mathematical relationship.

UNDERSTANDING THE *EQUALS* SIGN

The problem

The *equals* sign is used by students from the very first written mathematics that they do. But do they know clearly what it means? In their environment, both in and out of school, they see the *equals* sign used loosely as a sign of association (e.g., MATH = FUN) and in a causative sense (e.g., HARD WORK = SUCCESS). In elementary school arithmetic the *equals* sign is also often used with a sense of association or causation, instead of strict equality. Children who have spent years seeing questions such as

$$12 \infty 3 =$$

$$4 + 5 =$$

$$100 - 98 =$$

come to believe that = is a shorthand for "work this out now" and that it separates the question from the answer. For example, it is quite common for students to work out $3 \infty (14 + 36)$ by writing

$$14 + 36 = 50 \infty 3 = 150$$

just as they might push the buttons on a simple calculator:

$\boxed{1}$ $\boxed{4}$ $\boxed{+}$ $\boxed{3}$ $\boxed{6}$ $\boxed{=}$ [50 on screen] $\boxed{\infty}$ $\boxed{3}$ $\boxed{=}$ [150 on screen].

The problem with using the *equals* sign in this way is that, when the statement is looked at as a whole, it says that $14 + 36 = 150$, which is not true.

When they begin to solve algebraic equations, students often set out their work like this:

$$2x + 6 = 17$$

$$= 2x = 11$$

$$= x = 5.5$$

Although this gives the correct solution, the *equals* signs on the extreme left of the second and third lines should not be there. They are not being used to join equal things (17 is not equal to $2x$, 11 is not equal to x, and certainly 17, 11 and 5.5 are not all equal!). The student is probably using the *equals* sign to join a question ($2x + 6 = 17$) and an answer (what you get when you work it out: first $2x = 11$ and then $x = 5.5$). In algebra, this interpretation of = is not appropriate. There usually is no "question" on one side of the *equals* sign and no "answer" on the other. Instead, students have to deal with chains of equality and with logical consequences of whole statements of equality, knowing, for example, that

if $100 = 40 + 60$

then $200 = 80 + 120$

To solve equations, students have to write chains of logically equivalent statements of equality that lead them towards a solution. For example,

if $a^2 + 6a + 8 = 0$

then $a^2 + 6a + 9 - 1 = 0$

and so $(a + 3)^2 = 1$

Some suggestions

- Write statements of equality in a variety of ways. For example, instead of always writing 3 ∞ 4 = □ , occasionally try □ = 3 ∞ 4.
- Make "mats" out of cuisenaire rods or other material and write the resulting equalities in many ways. For example, the 12-mat shown in Figure 5 could lead to equations such as

  12 = 3 ∞ 4

  and

  7 + 4 + 1 = 4 + 4 + 4 = 3 ∞ 4 = 10 + 2, etc.

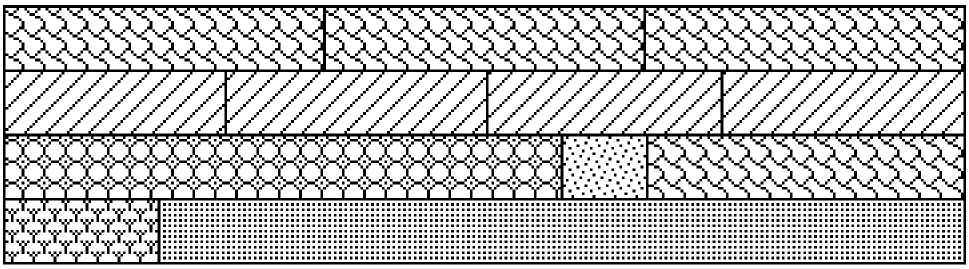


Figure 5: Cuisenaire mat showing ways of making 12

Older students can be challenged by an activity like this if the value given to the white unit rod is not one, but something like 0.1 or one third. Solving equations by a guess-check-improve strategy can be introduced as a puzzle if one of the numbers is hidden, e.g.,

$7 + \square + 1 = 10 + 2$ or $7 + \square + 1 = 3 ∞ \square$

- Don't accept misuse of the *equals* sign, when students link unequal expressions as in 14 + 36 = 50 ∞ 3 = 150. Suggest that they use arrows instead, or a personal invented notation without a standard meaning.

UNDERSTANDING PROPERTIES OF NUMBERS

The problem

One of the most frequent algebraic acts is manipulation - changing an expression into an equivalent expression that has the same value; for example, replacing 3x + 6 by 3 ∞ (x + 2). Students learn rules about how this is to be done, but they very frequently forget them. When they forget, they must be able to go back to numbers to help them recollect. For example, we watched a Grade 10 boy while he was trying to solve the complicated equation:

$$\frac{f-5}{2} + 4f = 2$$

He decided to multiply the fraction $\frac{f-5}{2}$ by 2. He was unsure how to do this so we suggested he should think how he would multiply the fraction $\frac{3}{4}$ by 4 as a guide to the procedure. This did not help the boy at all, because he thought that he should multiply both the top and the bottom of the fraction, giving $\frac{12}{16}$. This boy should have been guided in his algebra by his knowledge of number, but this knowledge wasn't strong enough. Difficulties like this are very common in algebra classes.

Some Suggestions

- There are two main aims of number pattern work in elementary school: firstly, to develop facility, flexibility and familiarity with numbers, and secondly, to build understanding of their general properties. The second aim, which is so important for students learning algebra, seems to us to need more emphasis. Many students will show their intuitive understanding of the properties when they do arithmetic. Activities such as those in Figures 6, 7 and 8 will develop this informal understanding. When they have finished activities like these, students need to be encouraged to explain what they have discovered and why they think it would always be true.

INVESTIGATING PATTERNS (i)

Investigate answer patterns associated with an operation, e.g.,

16	26	36	46
-9	-9	-9	-9

and predict answers to related questions, e.g., 86 - 9.

(Australian Education Council, 1991, p. 192)

Figure 6: A pattern that is easy to see but hard to say

The pattern involved in the question in Figure 6 is easy to see, easy to use and easy to demonstrate. However, it is hard to verbalise without technical language. If we start with ten more objects before the subtraction but take away the same amount, we will still have ten more after the subtraction. In later years, it can be expressed succinctly, using algebra, as

$$(a + 10) - b = (a - b) + 10$$

and more generally as

$$(a + c) - b = (a - b) + c$$

Students need an intuitive familiarity with number properties such as this.

INVESTIGATING PATTERNS (ii)

Investigate what happens to the sum and the product when we increase one number and decrease the other:

9 + 9 = 18, 10 + 8 = 18, 11 + 7 = 18, but

9 ∞ 9 = 81, 10 ∞ 8 = 80, 11 ∞ 7 = 77, ...

Use the pattern to predict answers to multiplications such as

21 ∞ 19, 22 ∞ 18, ... (from 20 ∞ 20 = 400), and

101 ∞ 99, 102 ∞ 98, ...(from 100 ∞ 100 = 10000).

Use your calculator to check the predictions.

(based on NCTM, 1989, p. 43)

Figure 7: Adding and multiplying give different patterns

Figure 7 shows an investigation that develops insight into the relationships between operations. Reducing one number and increasing the other makes no difference to the sum, but affects the results of multiplying in an interesting way.

FRACTION SEQUENCES

Take any fraction (e.g. $\frac{1}{4}$). Make a new fraction by adding 1 to the numerator and 2 to the denominator. Work out its decimal value on a calculator. Record the fraction and its decimal value in a table. Make another fraction by adding 1 to the new numerator and 2 to the new denominator. Work out and record its decimal value. Continue. What happens? Why? It is interesting to plot the decimal values on a number line (from 0 to 1).

0 0.25 0.375 0.5

Example:

Fraction sequence $\frac{1}{4}$ $\frac{2}{6}$ $\frac{3}{8}$ $\frac{4}{10}$ $\frac{5}{12}$ $\frac{6}{14}$ $\frac{7}{16}$

Decimal sequence 0.25 0.33 0.375 0.4 0.416 0.428 0.437

Start with other fractions and see if a similar thing happens. Make up other rules, such as adding 2 to the numerator and 3 to the denominator. What happens? Why?

Figure 8: A problem linking fractions and decimals

The activity outlined in Figure 8 is very rich. It makes links between decimals and fractions. It also relates the fraction $\frac{a}{b}$ to the result of the division $a \div b$, something that many students never notice and are not shown. After or during the investigation, the teacher should ask whether fractions can be added by adding the numerators and the denominators. Let's hope the answer is no!

MULTIPLES

Have students make an addition square and a multiplication square (say up to 12 ∞ 12 or larger) and colour in all the numbers which are multiples of 2. Find a rule for predicting which numbers will be coloured in and which will not. (Note: printed multiplication and addition squares are useful here)

Further investigations: Using fresh squares, colour in all the numbers that are multiples of 3 (then 4, 5, etc.). Find a rule for predicting which numbers will be coloured in and which will not.

Figure 9: A problem about factors and primes

With the activity outlined in Figure 9, students will find that the patterns coloured in the multiplication and the addition squares are different. They will observe that a number in the multiplication square is a multiple of 2 if either one of its factors is a multiple of 2. (They will need to relate even numbers to multiples of two.) In the further investigations they will find that similar rules hold for multiples of 2, 3 or 5, but not for multiples of 4. Why is this?

USING ALL NUMBERS, NOT JUST WHOLE NUMBERS

The problem

One of the big transitions to be made over the middle grades is to move students towards dealing confidently with the complete real number system which includes fractions and decimals and negatives. Algebra requires students to be able to work with all these numbers. Historically, negative numbers came to be accepted in mathematics because they arose as intermediate steps in finding positive solutions to algebraic equations. Previously they had been rejected because there are no negative quantities in physical reality. Getting students to operate easily with fractions and decimals and negatives is clearly a big task. Some suggestions for activities that are particularly related to algebra are given below.

Some suggestions

- Experiment with calculators. Most students know that the order in which addition or multiplication is carried out does not matter with whole numbers, for example that 3 ∞ 5 = 5 ∞ 3 and 7 + 9 = 9 + 7. This knowledge of commutativity halves the effort required to learn multiplication tables and addition facts. However, many students do not know whether the same property holds for fractions, decimals or negative numbers. For example, is 3.7 ∞ 4.6 equal to 4.6 ∞ 3.7?

Students perceive that they have to carry out calculations differently with whole numbers and fractions and decimals. For example, the way to set out an addition of fractions and the procedures to use (e.g., finding a common denominator) are very different from the ways to set out an addition of decimals or whole numbers. How can they know whether the basic number properties apply to all types of numbers? This needs explicit attention. Calculators can be used to generate large numbers of examples quickly. Students who work with arrays of dots or areas to convince themselves of the commutative property (see Fig. 10) need to talk about whether the same property would hold for decimals or fractions and how they might demonstrate that it does. An area model (see Fig. 10) can be used for this, but it is not easy for many students.

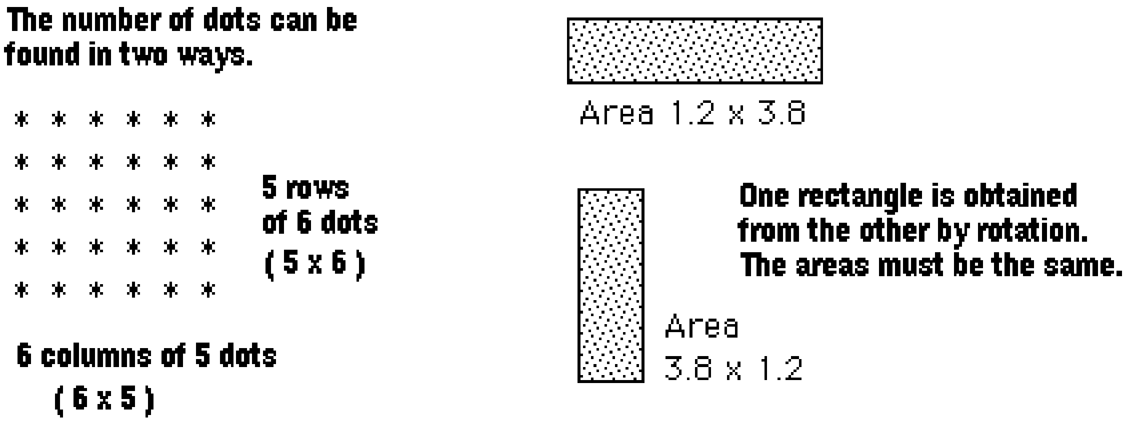


Figure 10. The dot model is simpler than the area model but only works for whole numbers.

- Most students know that division and multiplication are inverse operations and that addition and subtraction are inverse operations. This important knowledge can be reinforced in a variety of ways. For example, students can learn addition and subtraction facts such as 7 + 4 = 11, 11 - 7 = 4 and 11 - 4 = 7 together. They can play simple "guess my number" games (e.g., I am thinking of a number, I multiply it by 5, I add one. The answer is 21. What was my original number?) However, many students will not know whether the inverse operation property still holds when the numbers are not whole numbers. A 12-year old girl, for example, was convinced that multiplication could be "undone" by division but was not sure whether division was always "undone" by multiplication. She said that for some numbers like 18, division is "undone" by multiplication (18 ÷ 3 ∞ 3 = 18) but that it is not undone for numbers like 16. She explained that 16 ÷ 3 is "5 remainder 1", but "5 remainder 1" multiplied by 3 is not 16. To operate successfully with algebra she needed to move beyond the realm of whole numbers and into the full range of real numbers.

- When students are working with the pattern shown in Figure 7, observe where they stop (see Fig. 11). Do they stop at 17 and 1, like Ann, or do they go on to 18 and 0, like Ben? Do they realise that the pattern can be continued indefinitely, as shown in Con's work? And what about fractions? How do

  $$10\frac{1}{2} + 7\frac{1}{2} = 18 \quad \text{and} \quad 10\frac{1}{2} \infty 7\frac{1}{2} = 78\frac{3}{4}$$

  fit the pattern?

Ann's work

	+	∞
9,9	18	81
10,8	18	80
11,7	18	77
12,6	18	72
...	...	...
16,2	18	32
17,1	18	17

Ben's work

	+	∞
9,9	18	81
10,8	18	80
11,7	18	77
12,6	18	72
...	...	...
16,2	18	32
17,1	18	17
18,0	18	0

Con's work

	+	∞
9,9	18	81
10,8	18	80
11,7	18	77
12,6	18	72
...	...	...
16,2	18	32
17,1	18	17
18,0	18	0
19,-1	18	-19
20,-2	18	-40
21,-3	18	-63
etc.		

Figure 11: Extending number patterns

WORKING WITHOUT A PRACTICAL CONTEXT

The problem

In early grades students learn the commutative property for addition and multiplication, i.e., that the order in which two numbers are added or multiplied doesn't matter. Many students might also accidentally infer that the same is true for subtraction and division. Because in the elementary school they only ever take smaller numbers from larger numbers or divide a smaller number into a larger, they learn to ignore the order. They never have to do 5 - 25 or 5 ÷ 25. To solve a problem they rely on what it is about to help them decide which order for calculating will give a likely answer. In algebra, there is frequently no "context" to guide students as to which way to write subtractions and divisions; they must choose deliberately between $m - n$ and $n - m$ without clues from the size of the numbers involved.

Some suggestions

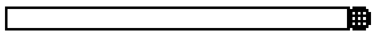
- Practice with questions such as those in Figure 12 may help. Questions structured like the first part of KIM'S AGE and the last part of MATCH are often misinterpreted by students of all ages. The clues to meaning in mathematical text will often come from the grammar and word order, not from the context or from a superficial reading of key words like "younger" and "shorter". Working out the meaning requires a different, slower and more careful style of reading than that used for ordinary text.

KIM'S AGE

Kim is 16. Kim is 4 years younger than Al. How old is Al?

Kim is 16. Greg is 4 years younger than Kim. How old is Greg?

MATCH



(i) Draw a line twice as long as the match.

(ii) Draw a line 1 cm shorter than the match.

(iii) The match is 2 cm shorter than a line. Draw the line.

Figure 12: Questions to promote careful reading

CONCLUSION

The activities we have described in this article illustrate some of the aspects of arithmetic that students need to securely understand in order to learn algebra well. The particular selection of ideas has been guided by our observations of students' difficulties in using algebraic notation and solving equations - the major topics of introductory algebra in most schools. Teachers will find further articles and a series of workshop activities on the development of algebraic concepts and skills in *MathsWorks: Pattern, Order and Algebra* (MacGregor et al.,1994). The *Curriculum and Evaluation Standards* (NCTM, 1989) also contains many activities to support the development of algebra.

Algebra is that part of mathematical language which has been designed to express generality. For this reason, it is the part of mathematics where an understanding of the general properties of numbers and the relationships between them is most crucial.

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