Origins of Students' Errors in Writing Equations

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A set of test items was designed to eliminate as far as possible the factors believed to cause the errors in the formulation of simple linear equations. The test items were given to 281 year 9 students in seven schools. The high incidence of errors in the students' responses could not be attributed to the causes widely documented in the literature. A theory of cognitive models was proposed to account for the errors as well as for the various forms of correct equations. According to the theory, cognitive models of relations between two variables represent contrast between large and small entities and do not conform with the structure of algebraic notation. Students frequently make errors when they try to represent these cognitive models directly.

In spite of changes in the mathematics curriculum and in mathematics teaching practices, students continue to misunderstand, to forget, and to make errors. It is generally accepted in the research community that many errors are not accidental mistakes or random guesses; students try to make sense of mathematical tasks, and can often explain and give reasons for their wrong answers. Many studies of the mathematical thinking underlying errors have involved interviews or tutoring sessions with individual students. The analysis of what students say in these sessions has provided insight into reasons for misconceptions and errors. However there is debate (Ginsberg, Kossan, Schwarz & Swanson, 1983; Kirshner, Awtry, McDonald & Gray, 1991) as to whether "think-aloud" sessions or introspection can reveal what is really happening at deep levels of comprehension and reasoning. In the study described in this paper we obtained clues to students' thinking by analysing their written responses to test items. In at least one important respect, the written responses revealed different thinking to that generally proposed by researchers who have used interview methods.

This paper proposes a theory of the cognitive processes underlying students' algebraic behaviour. The theory provides an explanation of students' difficulties in writing simple equations and suggests that it is important to teach students that an algebraic formulation of an equation often does not directly represent the most salient features of the relations involved.

Errors in writing simple linear equations

There are many published reports of students' errors in writing simple algebraic equations (e.g., Clement, Lochhead & Monk, 1981; Cooper, 1984; Kaput & Sims-Knight, 1983; Mestre, 1988). A complete discussion of the literature is given by MacGregor and Stacey (to apear). In summary, it is widely accepted that students make errors because of:

1. The use of algebraic letters as abbreviated words (e.g., *a* means "apple", not "number of apples").
2. Attempts to translate directly from key words to mathematical symbols, from left to right, without concern for meaning
(e.g., "There are six times as many cats as dogs" is translated incorrectly
as 6 x *c* = *d*).
3. Use of the "equals" sign to indicate that what is on the left is loosely associated with what is on the right (e.g., 20*p* = *t* could mean "There are 20 pupils for every teacher").
4. The misleading influence of mental pictures (e.g., groups of 20 pupils and individual teachers seen in the mind's eye, and represented on paper as 20*p* + *t*, 20*p* = *t* or 20*p* : *t*).

The type of error shown in the examples above, where the numeral is associated with the wrong variable in a simple linear equation, is referred to in the literature as the *reversal error*. It is currently accepted (Herscovics, 1989; Laborde, 1990; Mestre, 1988) that a major cause of reversal error is the attempt to translate directly from words to symbols, as described above in #2.

There are several reports of interviews with individual students (Clement, 1982; Clement, Narode & Rosnick, 1981; Sims-Knight & Kaput, 1983) about their reasons for making the reversal error, and of tutoring sessions (Cooper, 1984; Rosnick & Clement, 1980; Wollman, 1983) designed to overcome it. Students' resistance to accepting that their reversed equations are wrong (Rosnick et al., 1980), and their feelings of uncertainty when they are correct (Clement, 1982; MacGregor, 1991), suggest that the error has a deep-seated cognitive origin. The results of the present study show that the published causes listed above do not adequately explain the reversal error and they also provide clues to students' patterns of thinking about questions of this type.

Method

Test items were prepared to eliminate the documented sources of reversal error and to reduce as far as possible the incidence of errors from other causes. Three of the items were designed to be straightforward translation tasks, expected to be easy for students to get correct. These items were:

A. "*z* is equal to the sum of 3 and *y*." Write this information in mathematical symbols.

B. "The number *y* is eight times the number *z*. " Write this information in mathematical symbols.

C. The Niger River in Africa is *y* metres long. The Rhine in Europe is *z* metres long. The Niger is three times as long as the Rhine. Write an equation which shows how *y* is related to *z.*

The items are grammatically very simple, unlike some of the translation tasks reported in the literature. With the possible exception of item C, the algebraic letters have no association with concrete objects and hence cannot be confused with the initial letters of names. The numerals would not be perceived as adjectival modifiers of nouns (Kaput et al., 1983), in the way that "3*a*" might be read as "three apples". Even in the last item, it seems unlikely that students would think of "3*z*" as meaning "three Rhines". To deal with the first item, students need to know that "the sum of 3 and *y*" is written as "3 + *y*" or "*y* + 3". In the other two items, direct translation from key words to mathematical symbols, in order from left to right, would yield correct equations.

The items were included in a test for year-9 students (average age 14) at the end of the third year in which some algebra is taught, and the test was used by teachers in seven schools for their own classes. To enable us to estimate the incidence of poor English comprehension or difficulties in arithmetic, the test included simple non-algebraic items such as:

At a meeting there are five more women than men. There are 25 women.

How many men are there?

About 5% of the sample were found to have difficulty with this and similar items. We assumed that the other 95% could read adequately, could understand the phrases "more than" and "times as long as", and knew when to add and when to multiply. They were expected to make few if any errors in the three target items.

Results and Discussion

Although the published causes of error had been eliminated as far as possible from the target items, many students made mistakes. As shown in Table 1, approximately one-half were correct for Item A, and only about one-third were correct for items B and C. Most wrong responses to these two items showed reversal error: that is, the numeral was incorrectly associated with the larger variable. For example, a common response to item B was $8y = z$, instead of the correct $y = 8z$. Similarly for item C, many students wrote $3y = z$, $3y + z$ or $3y \times z$ instead of the correct $y = 3z$.

Table 1

Percentage of students correct and incorrect on items A, B and C

Item	*n*	Correct	Incorrect	No response
A	255	57%	39%	4%
B	281	37%	57%	6%
C	281	30%	53%	17%

Reversal error was rare in responses to item A. As may be seen in Table 2, the majority of wrong responses for that item were $z = 3y$ and $3y = z$, both showing the well-documented error of writing numeral and variable side by side to indicate a sum. What is of more interest in this item, however, is that it gives a clue to whether students were using direct left to right translation of key words to symbols. A direct translation procedure for item A may have been used by the 107 students who wrote $z = 3 + y$ or $z = 3y$, matching the order of words, but a similar number of students gave responses which could not have been produced in this way: 66 students wrote $3 + y = z$, 9 wrote $y + 3 = z$, and 36 wrote $3y = z$. These three forms, having "*z*" after the equals sign, could not have been produced by a direct left to right translation of key words to symbols. If direct translation was a common procedure, we would have expected more students to have produced responses matching word-order than not. It therefore appears that information obtained by reading the item had been mentally transformed or rearranged before being represented in mathematical symbols. The equations came from a cognitive model, not from direct unprocessed translation of the sentence. This observation is a clue to the cause of reversal in the other two items.

Table 2

Responses to item A (*N* = 255)

Response	Frequency
$z = 3 + y$	70
$z = 3y$	37
$3 + y = z$	66
$y + 3 = z$	9
$3y = z$	36
Other[a]	27
No response	10

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[a] The category "Other" includes responses such as *zy*, $z = y$, $z^3 = y$ and $y^3 = z$.

From words to mathematical representations

Although direct substitution of mathematical symbols for words in item B would have produced a correct equation, the forms of students' responses show that no more than 26% of them could have been produced by this method. Over one-half (56%) were reversed equations or expressions such as $8y = z$, $z = 8y$ or $8y \times z$. These are not translations of either the given sentence or a valid paraphrase of it.

Moreover, students who wrote correct equations were almost as likely to write $8z = y$ (44 responses) as $y = 8z$ (50 responses). Similar apparently random selection of equation format for item A has been referred to above. It was also evident in responses to item C, where correct equations were $3z = y$ (43 responses) and $y = 3z$ (34 responses). This finding, consistent for all items, is evidence that surface word-order and syntax are not necessarily represented at the level of comprehension, and supports the notion that students write equations by drawing on a non-linguistic conceptual model of meaning.

Mental representations of compared quantities

It has been postulated by linguists and psychologists that comprehension of spoken and written language takes place in a non-linguistic cognitive system. Johnson-Laird (1983) has proposed that this cognitive system does not involve the logical form of a proposition or the syntax of its expression in natural language; comprehension and reasoning are based on a mental model of perceived semantic features.

The finding that the order of the items in the algebraic representations produced by students was apparently random, suggests that they were drawing information from a non-linguistic cognitive representation in which the arrangement of components was arbitrary. This conjecture is supported by the work of Das, Kirby and Jarman (1979) and Stassen (1985) who propose that information in a mental model of compared quantities (such as the sizes of the variables in items A and B or the lengths of the rivers in item C) is not ordered in time or space. The model is set up automatically when the question is understood and thereafter can be used for a variety of inferences and expressions. No syntactic or logical rules are required to make these inferences, which are directly "read off" the model. In the case of item A, for example, the model of two equal entities - one identified as *z* and the other a combination of *y* with 3 - can be represented with any of these three symbols written first. As shown in Table 2, some students wrote *z* first, some wrote 3 first, and some wrote *y* first. The other two items are about unequal quantities. In item B, *y* is larger than *z*; and in item C the Niger River is longer than the Rhine. Approximately one-half the responses were reversals (52% and 49% respectively). According to our theory, semantic mental models of these two situations represent contrasted large and small entities, and not situations of equality. In this respect, the underlying model constructed by the student does not conform with the structure of equations. Although the models represent meaning and support comprehension and reasoning, they cannot be represented directly in mathematical symbolism. Students who try to represent the structure of their semantic models write reversed equations or expressions in which the numeral is associated with the larger variable. A more complete discussion of these ideas is given by MacGregor and Stacey (to appear).

According to Kirshner et al. (1991), the successful translator first attempts and then may reject a direct translation from words to symbols, whereas unsuccessful translation is caused by missing grammatical cues to abandon simple translation. However, for our items, using direct translation would have resulted in success whereas rejecting it led to failure. Moreover, our data support thte conclusion that direct translation was not commonly attempted. In this respect, our results contrast starkly with the current view in the literature that mistakes are generally due to direct left to right translation without regard for meaning. We suggest that in studies whih have used interview methodology, students may have used direct translation *post hoc* as a means of justifying an incorrect response in an interview, rather than as the cognitive process which initially produced the incorrect response.

Conclusion

In general, students have no difficulty comprehending the items used in reversal error studies (Wollman, 1983). Major difficulties in translating from brief natural language statements to algebraic equations are not primarily due to incorrect concepts or misunderstanding of meaning (Kirshner et al., 1991). The data presented in this paper indicate that a major source of students' errors may be their belief that mathematical symbols can be used in an intuitive way to express directly what they understand about a situation.

Students' written expressions in elementary algebra can provide clues to the "inarticulated mental processes" (Clement et al., 1981) that are not available for conscious access and verbal description. The theory of semantic models that we propose is in accord with psycholinguistic theories postulating the independence of semantic and syntactic processing and the structure of semantic models. It offers an explanation of students' difficulties in using algebra to express relations between variables.

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