ALGEBRAIC SUMS AND PRODUCTS: STUDENTS' CONCEPTS AND SYMBOLISM

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This paper presents evidence on the difficulties that students have in symbolising sums and products algebraically. A cluster of obstacles related to the conjoining error have been identified in the literature and are seen by many to be the principal causes of students' notational difficulties in elementary algebra. This paper reports on the incidence of the conjoining error in a variety of new situations. The incidence of the error was found to be very variable. It was low in solving linear equations and did not arise at all in formulating an equation. Many of the incorrect responses are explained not by the published obstacles, but by examining the non-mathematical parallels which students draw on to write algebraic expressions. At all levels there were students who did not make clear distinctions between addition, repeated addition, multiplication and exponentiation.

It is well known that the algebraic symbolism for algebraic sums and products is puzzling for beginners, who must learn that the sum of x and y can only be written as $x+y$ or $y+x$ whereas the product can be written as something other than $x \infty y$. A textbook popular in the 1950's, for example, warns students that the notation for algebraic sums is their "first difficulty". This paper explores how students represent and interpret algebraic sums and products in a variety of situations and examines the adequacy of current explanations for students' errors.

Writing conjoined terms for sums

Many research studies have documented the tendency for students to use the notation for algebraic product for the algebraic sum, i.e., to use the conjoined expression ab where $a+b$ is appropriate. Kuchemann (1981), working with the CSMS project, found that the error was extremely prevalent, and Booth (1984), after investigating the underlying causes of the errors that the project had exposed, concluded that conjoining in algebraic addition was a serious problem. The prevalence of the error in British students was confirmed by the monitoring of the Assessment of Performance Unit (1985). Examples of the items used in these British studies, showing the very high rates of conjoining errors on some items, are given in Table 1. These authors see the conjoining error as a significant error which is hard to remediate and with an important cause, as discussed below.

As with other common errors and interpretations in mathematics, there are several different reasons why students may believe that $a+b$ should be written as ab. Kuchemann (1981) noted that students who made errors derived from this misunderstanding (e.g., writing $8ab$ as a simplification of $2a+5b+a$) were generally at the lowest level of understanding on the CSMS test. He saw this

[1]This research was funded by a grant from the Australian Research Council to Kaye Stacey for the project "The Cognitive and Linguistic Aspects of Learning to Use Algebra". We wish to thank the teachers and students who participated in the testing for their valuable assistance.

error as an important indicator of "acceptance of lack of closure", a characteristic which Collis had identified as developing through the stages of cognitive growth. For Collis, an individual requires different levels of closure during the progression from concrete to formal thinking. In the early stages, children can only work on tasks where each expression can be evaluated. In Collis' later work (Collis & Biggs, 1979) acceptance of lack of closure is used in a wider sense, noting a mature student's tendency not to restrict an answer, but to generalise from the given, making links with new data. In the psychology literature, "closure" refers to the completion of a Gestalt when parts of the stimulus are missing, in order to make sense of it.

Kuchemann (1981) and Booth (1986) also observed that many students may write 7*n* as the answer to the question "Add 4 onto 3*n*" because they believe the expression 4*n*+3 is not acceptable as an answer. In the terms of Sfard (1991), students see the expression as representing the process of adding, not the object that results from the adding. Tall and Thomas (1991) have summarised these observations by identifying a cluster of obstacles involved with the conjoining error: those related to parsing, process/product confusion, lack of closure and the nature of the expected answer.

From a constructivist perspective, it is noted that students will draw on previous and concurrent learning from other areas to work with algebraic symbols. They will make parallels with other notation systems, such as in writing fractions where conjoining represents addition (e.g., $3\frac{1}{2} = 3+\frac{1}{2}$), in chemistry where CO_2 is produced by adding oxygen to carbon, and in music where ♩♪ lasts for one and a half beats. Chalouh and Herscovics (1988) observed Grade 6 and 7 students interpreting algebraic expressions in terms of other frames of reference (e.g., place value 53 = 50+3).

Booth (1984) and others use the words "closed" and " unclosed" to distinguish answers such as 7*n* and 3*n*+4. However, in this paper we refer instead to "conjoined terms" (for 7*n*, etc.) because "closed" does not distinguish between observed responses and proposed causes.

The review above shows that many researchers believe that errors based on conjoining are particularly significant indicators of cognitive growth. There is, however, some disagreement. Sutherland (1991) has found that children working with Logo and spreadsheets accept "unclosed" expressions such as *x*+7 without difficulty and she questions the claim that the need for closure is a major obstacle in learning algebra. Tall and Thomas (1991), also working in a computer environment, noted that there needs to be "a reassessment of fondly held beliefs of what is hard and what is easy" (p. 145).

In this paper we present data which extends information on the prevalence of conjoining errors. Previous studies used only items requiring simplifying or formulating algebraic expressions. We present further data on similar items but we also look for evidence of the conjoining error in students' attempts to formulate and solve equations. We examine the data to see to what extent students use conjoining only for addition and we examine the adequacy of the published cognitive obstacles as causes of students' responses.

Method

This paper brings together data from a number of separate studies. Tests which included the items used in this paper were constructed and marked by the authors and administered by volunteer teachers in 24 Australian secondary schools in 1991-93. Although the participating schools were not randomly selected, the sample includes schools from two states, from all socio-economic areas and a mix of government and private schools. Almost all of the participating schools take students of all abilities from the local area and teach them in mixed-ability mathematics classes. In order to expose important difficulties that are widespread, the results have been analysed by class and by school as well as by individual as is reported in this paper and were found to be reasonably consistent across classes. In this way we hope to reduce the limitations of this form of sampling to be confident that the results are not confined to any particular school, category of school or textbook. The tests contained items testing ordinary reading competence, the results of which established that literacy in English was a limiting factor for no more than 3% of the students involved.

Table 1: The prevalence of the conjoining error in published studies

Source	Item	Percentage correct	Percentage who conjoined for addition and used exponents for multiplying
Kuchemann 1981 Item 13(iv)	Simplify $2a+5b+a$	60% (14 yr olds)	20% ($8ab$)
Kuchemann 1981 Item 4(ii)	Add 4 onto $3n$	36% (14 yr olds)	31% ($7n$)
APU 1985 Item A4	Peter's age is represented by x. Alan is 2 years older than Peter. How can we represent Alan's age?	49% (15 yr olds)	10% ($2x$)
APU 1985 Item A5b	How do we represent the number which is three less than n?	45% (15 yr olds)	3% ($-3n$)
APU 1985 Item A5c	How do we represent the number which is twice n?	44% (15 yr olds)	23% (n^2)
APU 1985 Item A6	I have x pence and you have y pence. How many pence do we have altogether?	48% (15 yr olds)	34% (xy)
APU 1985 Item A7	I have x pence and you have *3* pence. How many pence do we have altogether?	46% (15 yr olds)	36% ($3x$)
APU 1985 Item A8	A bar of chocolate costs x pence and a packet of crisps costs y pence. What is the cost of 2 bars of chocolate and 3 packets of crisps?	59% (15 yr olds)	9% ($5xy$, $2x3y$)

Results and Discussion

The items used are shown in Figure 1. The number of students for each item and the number of schools from which they came is given in Table 2 along with the percentage correct and making conjoining errors. The final column also records the percentage of students who used exponential notation to represent multiplication. Because of space limitations, the table gives results for students in Years 7 and 10 only. The students in this sample have performed similarly to students in the APU samples on some items, but the overall rates of conjoining are markedly lower.

Item MS1. Write in mathematical symbols:
(i) Add twelve to *x*. ....
(ii) Multiply *x* by three. ....

Item MS2. Write the following in mathematical symbols:
"Add 5 to an unknown number *n*, then multiply the result by 3".

Item MS3.
(i) David is 10 cm taller than Con. Con is *h* cm tall. What can you write for David's height?
(ii) Sue weighs 1 kg less than Chris. Chris weighs *y* kg. What can you write for Sue's weight?
(iii) Tina has twice as much money as Dino. Dino has $*n*. What can you write for the amount of money Tina has?

Item MS4. Which of the following expressions can be written as $x + x + x + x$?
(Circle one or more of the answers below)

$x + 4$ $x \infty 4$ $4x$ x^4 4^x

Item MS5. [Presented following table in vertical format, asked students to calculate three values, describe the rule in words and then asked them to "Use algebra to write a rule connecting *x* and *y*."

x 1	2	3	4	5	6	7	8	.	.	.	.
y 5	6	7	8	9	10	11	12	.	.	.	.

Item MS6. Solve the following equations [sample only]:
$4x - 7 = 3$, $16.4x - 7.9 = 0.3$, $34 - 8x = 10$, $2 + 0.6x = 2x$

Item MS7. Given $H=3f^2g$, find *f* in terms of the other letters. [Then similarly find *g*]

Figure 1: Items used in testing

1. Conjoining in translating into algebraic expressions

Item MS1 in Figure 1 requires students to translate statements in words directly into algebraic expressions. Table 2 shows that these items were correctly answered by a high percentage of students. A popular response for part(i), which was counted as correct in the table, was "$x + 12 =$ ". Explicit use of the multiplication sign ($3 \infty x$ or $3 \infty x =$) was the most common correct response for part (ii). The use of the "equals" sign suggests that students regard the expressions as incomplete and needing to be evaluated.

The very low incidence of conjoining in this item is surprising given the APU and CSMS results for Items A6 and 4(ii) shown in Table 1. Possibly Item MS1 asked more directly than did the British items that the procedure "add twelve", rather than the answer, be symbolised, .

The conjoined term was written for addition by only 24 individuals (3.5% of the total sample), but it was not confined only to younger students and it occurred in 17 of the 21 classes. Three of these students used conjoining again (now correctly) for the multiplication in part(ii) and four used division. However, fourteen used the multiplication sign explicitly and three used exponents. Thus about 3% (i.e., 14+3) of the total sample showed that they used conjoining for addition but not for multiplication.

Table 2: Success rates on items and incidence of conjoining

Item	Sample	% correct Yr 7 - Yr 10	% who conjoined for + or - Yr 7 - Yr 10	% who used exponents for x Yr 7 - Yr 10
Item MS1(i) Add 12 to x	678 students (9 schools)	68% - 93%	2% - 2% ($12x$)	
Item MS1(ii) Multiply x by 3	678 (9 schools)	73% - 91%		2% - 2% (x^3)
Item MS2 (n+5) x 3	1806 (18 schools)	14% - 47%	14% - 12% ($15x$, $5x$x3)	about 1%
Item MS3(i) David h+10	1463 (16 schools)	39% - 74%	14% - 5% ($10h$)	
Item MS3(ii) Sue y -1	1463 (16 schools)	36% - 64%	8% - 3% ($1y$) 1% - 1% ($-1y$)	
Item MS3(iii) Tina $2n$	1463 (16 schools)	38% - 65%	2% - 1% (nn)	17% - 14% (n^2)
Item MS4 $x+x+x+x$	1034 (12 schools)	47% - 60%		40% - 34% (x^4 or 4^x)
Item MS5 $y = x+4$	951 students (11 schools)	29% - 58%	0% - 0%	
Item MS6 Solving one variable equations	97 (3 schools, Yr 10 only)		3% ($2+0.6x = 2x$ $0.6x = x$)	
Item MS7 Transposing equation $H = 3f^2g$	48 (3 schools, Yr 10 only)		16% ($H\text{-}g = 3f^2$)	

2. Conjoining in a multiple step expression

Item MS2 had the same stem as Item MS1 but required students to use two operations. The most common error was the omission of brackets (11% at Year 10, 22% at Year 8). Conjoining was very prevalent, although still not as prevalent as in several of the APU or Kuchemann items in Table 1. Common conjoined answers were $15n$ (about 8%) and $5n$∞3 (about 4%), and another 4% were responses such as $3(5n)$, $5n^3$, $15n^3$, $8n$, $18n$. In another test this item had been replaced by an item identical to MS2 except that 5 was replaced by 14. Although there was a similar amount of conjoining overall, the relative frequency of responses was very different. Only 3 of the 517 responses were $42n$ (down from 8%) but $14n$∞3 was more common. Many features of an item determine students' responses, one being the size of the numbers and the ease of automatic response.

The incidence of conjoining for "add 5 to n" here was markedly higher than occurred in Item MS1(i), which was identical except for the second stage of multiplication by 3. Presumably many students felt the need to work out an answer before proceeding to multiply by 3. The higher incidence of conjoining in this item is in marked contrast to the conclusion reported in the APU survey. On the basis of comparison of the incidence of conjoining in Items A6, A7 and A8 (see Table 1), the APU report concluded that "when two operations are involved (multiplication and addition), fewer pupils represent addition by juxtaposition and more get such items right." (p 331). We did not find this.

3. Conjoining in formulating expressions

In Item MS3, students had to deduce which operation was required and write the answers algebraically. In the three parts of this item, results for our sample (especially the 15 year old Year 10 students) were similar to the results for the parallel APU items A4, A5b and A5c (including the number of omissions), although a little better. The incidences of conjoining for addition and subtraction were similar, and in both cases many students used exponents for multiplication.

However, less than 3% overall used conjoining only for addition. We consider that if students had clear concepts for the three operations involved and believed that conjoining is the convention for addition, then they would write $10h$ or $h10$ or 10C for DAVID, $1\text{-}y$, $y\text{-}1$, $\text{-}1y$ or similar for SUE and $2 \infty n$ or n^2 for TINA. There were 28 students (2%) who did this. An additional 8 students gave the answers $10h$, $1y$ and n^2 indicating that they wished to symbolise multiplication differently to addition. This percentage is similar to the estimate of consistent students from MS1.

Forty students (3%) wrote down the numbers and the letters without concern for the operations linking them. They all wrote $10h$, $1y$ and $2n$. Although their response to DAVID suggests that they have used conjoining for addition, the other two responses indicate that they have used it for subtraction and multiplication as well.

The variety of incorrect responses to this item indicates that students draw on different ideas about letters and abbreviations to construct their algebraic expressions. Year 10 students in the sample used in their algebraic expressions letters as abbreviations for words (Dh = h+10, where Dh stands for David's height, h = D-10, or C+10 = D); letters standing for procedures (h = +10 or h = -10); letters as objects and objects as letters (h = David-10, h = Con-10). As these examples from Year 10 show, many of the wrong answers were expressed as equations rather than expressions. The students who wrote $x\text{-}10 = h$, $h\text{-}10 = x$, and $h = h+10$ show the need students felt to have a second variable.

Item MS4 is included to investigate further the confusion between addition, repeated addition and multiplication and exponential notation, which was evident in MS3(iii)-TINA. Adding like terms is one of the first skills taught in school algebra, so the low success rate (see Table 2) is disappointing. However, the choice by 40% of Year 7 students and 32-34% of Year 8, 9 and 10 students of an exponential form supports the result of MS3(iii) and the APU Item A5c**.** The improvement from Year 7 to Year 10 was minimal.

4. Conjoining in formulating equations

Item MS5(iv) required students to write the equation $y = x+4$ (or equivalent) to describe a relationship evident in the table of two variables (see Figure 1). Although the success rate (see Table 2) was very low, and there was great variety in the answers, there was no conjoining. No student wrote $y = 4x$, $y=x4$, or simply $4x$. One wrote $4x+$ and another wrote $x=4y$. The absence of the conjoining error is surprising given the variety of answers that were produced. Examples included $1x=5y$, $x=y$, xy, $x+y$, $x+4y$, $x=y+4$, $xy+4$, $x=1y$, $x+5$, $(x=1,y=4)$, $x=1+4y$. In some of these examples, students are simply "connecting" x and y as instructed in the Item (e.g. xy, $x+y$). Some tell a story in abbreviated form (e.g., $x=1+4y$ may say that "you start with x equal to 1 and add 4 to get y", and $x+4y$ may say that "you take x and add 4 to get y"). Many of these responses constructed by students themselves are far from the syntactically well-formed expressions and equations that could demonstrate a conjoining error.

5. Conjoining in solving equations

A sample of 97 Year 10 students were given a total of 251 equation solving questions (Item MS6 in Figure 1) as part of another study and its pilot (Bell, MacGregor & Stacey, 1993). Their solutions were examined for evidence of interpreting conjoined terms as sums. Only seven such errors were made (e.g., deducing $0.6x = x$ from $2+0.6x = 2x$, or $x = 160$ from $13x = 173$), each by only one student and each in only one equation. Some of these errors are what Carry, Lewis & Bernard (1980) classify as the very common *deletion errors*. We conclude that conjoining errors are not common in familiar numerical equation solving (by Year 10students), a "microworld" where addition signs are common and routine procedures are often practised. However, in transposing the literal equation $H = 3f^2g$ (Item MS7), the conjoining error rate was high (see Table 2). This item was less familiar (indicated by a relatively high omission rate) and more abstract. Presumably these characteristics prompted the conjoining error.

Conclusions

Where parallel items were used, the results of this testing are similar to but a little better than the APU testing reported in Table 1. However, none of the items provoked conjoining to the same extent as the items reported there. There was substantial variability in the incidence of conjoining from item to item. The greatest incidence of conjoining came from the two step item (MS2) and the difficult transposing of an equation (Item MS7). In formulating an equation, there were no instances of conjoining errors at all. The students who were not correct very often wrote ill-formed equations (discussed further below).

For items MS1 and MS3, it was estimated that about 3% of the sample used conjoining to represent addition and only addition. About half of the students who used conjoining for addition used it for subtraction and multiplication as well. These students are simply putting together the numbers and symbols in the question to make an answer that looks like algebra. They are not applying consistent laws of logic to well distinguished concepts, but are probably achieving closure only in the sense of completion of a Gestalt.

Items MS3(iii) and MS4 revealed a substantial percentage of students using exponential notation for ordinary multiplication or repeated addition, and little improvement from Year 7 to 10. We suspect that this is due to the concepts of repeated addition, multiplication and repeated multiplication (exponentiation) not being sufficiently distinguished by students. This is a very important aspect of arithmetic that needs considerable emphasis in pre-algebra courses.

The literature reviewed above on students' use of algebraic symbolism has listed several obstacles related to the conjoining error. Students' responses to this testing have provided examples of all of these. However, many of the responses which students have constructed (e.g., to Items MS3 and MS5) are not explained by these obstacles. Instead, they seem to arise from students' drawing parallels with notation systems other than algebra. The published obstacles seek to explain students' errors from within mathematical systems and ways of thinking and symbolising and from an assumption that operations are clearly conceived, but students seem instead to often use a personal shorthand that only looks on the surface like algebra. For example, the conjoining error results in a syntactically well formed answer, but students' answers are often ill-formed. The personal shorthands draw on principles from a variety of other (informal) systems, such as the practice of abbreviating words and leaving out unimportant words when writing a note. Personal shorthands are not used consistently from question to question, nor is one symbol (or lack of symbol as in conjoining) kept for one meaning. Students often achieve closure in the sense of completion of a Gestalt, rather than applying a formal rule. Because their personal shorthands use the same surface features as does algebra, it is difficult for teachers to explain that the meaning is quite different.

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