The Use of Taught and Invented Methods of Subtraction.

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Abstract

This study investigated the methods used by 1370 students in Grades 7 to 12 for carrying out subtractions presented in vertical format. It was found that the percentage of students not using the algorithms taught at school rose exponentially from 1% at Grade 7 to 26% at Grade 12. Invented methods were used significantly more often by students with below average achievement in Mathematics than by others. However, students using invented methods were no more likely to make errors than those using taught algorithms. Older students were more likely to be correct than younger students, regardless of method. Use of invented methods was not associated with score on a lateral thinking test. It is concluded that students generally used invented methods when they had forgotten the taught algorithms. The results support the constructivist recommendation that students, particularly those of below average achievement, should be encouraged to explore and invent a variety of mental and written methods as well as, or instead of, practising an efficient pencil and paper algorithm.

The Use of Taught and Invented Methods of Subtraction.[1]

It is well known that people invent many different ways to subtract. Davis (1984), for example, describes the ingenious way an eight year old student calculated 64 - 28. He first subtracted 8 from 4 to give -4. Then he subtracted 20 from 60 to give 40. His final step was to take 4 from 40, obtaining the correct answer 36. This is just one example of a fascinating variety of idiosyncratic methods of subtraction that have been documented across the world. (Backman, 1978; Carraher & Schliemann, 1985; Carraher, Carraher & Schliemann, 1987; Davies, 1978; Ginsberg, 1977; Jones, 1973; Labinowicz, 1985; Marks, Hiatt and Neufeld, 1985; McIntosh, 1978; Plunkett, 1981, Trafton, 1978)

Although the use of idiosyncratic methods of subtraction by individuals has been extensively reported, there are few studies concerned with the use of methods of subtraction in school populations and no good data on how their use evolves. Only three relevant studies, all very small in scale, were located in the literature. The first is that of Jones (1973), who investigated the different methods 11 year old children used to give written answers to six arithmetic questions. One was: "What is the result of 83 - 26?" With this question he found that, in a sample of 75 students, one third used the algorithm they had been taught and two thirds used invented methods. Contrary results were reported by Matthews (1981), who interviewed four different groups of students: 40 eleven year olds, 16 young students who had just left school, 19 student nurses (average age 17) and 18 trainee bus conductors (average age 27). Nearly all of them used the decomposition algorithm (see Figure 1) that they had been taught at school. The third study is that of Petitto and Ginsburg (1982), who reported that of 14 students (average age

[1] The authors thank the staff and the students of Scotch College Senior School, Melbourne for their co-operation.

19 years), 6 used taught algorithms and 8 used their own invented methods. Because of the small sample sizes and the choice of populations, these studies give little indication about how the use of algorithms or invented methods generally evolves.

However, it is important to know the use that students make of the algorithms that they are taught in order to make wise decisions about the place of algorithms in the school curriculum. For most of this century, debate about teaching algorithms has been concerned primarily with selecting the best pencil-and-paper algorithms to be taught in schools. In the last fifteen years, the accessibility of calculators in the community and in schools has been changing the parameters of the debate. Society no longer highly prizes the ability to carry out arithmetic operations with pencil and paper neatly, accurately, and with speed. Without the constraint of perfecting speedy algorithmic skills, methods of teaching which match more closely how children understand and learn can be more freely explored.

Figure 1. Two taught methods of subtraction

Decomposition	Equal Additions
Error!	$2\ {}^{1}4\ {}^{1}6$
$-1\ 5\ 9$	$-1_1\ 5_1\ 9$
$8\ 7$	$8\ 7$

For Decomposition, the number 246 is renamed in two steps as (1 hundred + 13 tens + 16 ones) and the subtraction is then carried out, column by column. For Equal Additions, the subtraction of 159 from 246 is replaced in two steps by the equivalent subtraction of (2 hundreds + 6 tens + 9 ones) from (2 hundreds + 14 tens + 16 ones) obtained by adding equal amounts to both numbers.

Even before the advent of the calculator, there was interest in flexible ways of performing arithmetic processes (see, for example, Jones, 1973). Several writers have claimed that following standard algorithms is a hard way to learn to perform calculations. Bell, Costello and Kuchemann (1983), for example, comment that:

> "the putting together of skill elements to produce an *ad hoc* method for the solution of a given problem is a more natural way of operating for children than the learning of a standard procedure....We are forced to conclude that this is not a mode of functioning for which the human brain is well designed; in this respect it is unlike the calculator or the computer". (Bell *et al*,1983, p. 93)

Flexible methods, derived by students themselves, are now being recognised in official curriculum advice. Recent publications in Australia (Ministry of Education, 1988; Australian Education Council, 1991) advocate that children invent and use a variety of methods, both mental and pencil and paper, as well as using calculators. In the United States, the NTCM Curriculum and Evaluation Standards (1989) has a similar emphasis. It states that students should be able to generate new procedures and modify old ones in the solution of mathematical problems. A subtraction, 75 - 26 is given as an example; students are expected to change the problem so that regrouping is not necessary. Children are also to be encouraged to invent alternative strategies for computing mentally (p. 45) and this is seen as a natural consequence of adopting a constructivist view of learning (Maher & Alston, 1990). Debate is just beginning about whether formal algorithms should be taught at all in schools.

The present study examined how the general pattern of algorithm use evolves. Subtraction was selected for the study because of the richness of idiosyncratic methods, although it is likely that the use of other algorithms may evolve in a somewhat similar way. The particular aims of the study were to establish the incidence of use of taught algorithms or invented methods of subtraction in secondary school students and to relate the use of invented methods to age, mathematical achievement and lateral thinking ability. Are taught algorithms generally used or gradually forgotten? Are they used more frequently by students of above or below average achievement? Are they associated with correct or incorrect answers? Are invented methods used by creative students? Since many of the reported invented methods seemed highly ingenious, it is possible that they may generally be used by creative individuals, who are good "lateral thinkers". In defining lateral thinking, de Bono (1970) states:

> "The most basic principle of lateral thinking is that any one particular way of looking at things is one from among many other possible ways. Lateral thinking is concerned with exploring these other ways by restructuring and rearranging the information that is available." (de Bono p. 63.)

He comments that "lateral thinking is also concerned with breaking out of the prison of old ideas." (de Bono 1970 p.11). This characteristic may therefore be associated with the use of invented methods of calculation.

Method

Methods Used for Subtraction.

Four subtraction questions (see Figure 2) were administered by the first author to students attending an academically oriented private school for boys. One thousand three hundred and seventy students (98.5% of the school population) were finally included in the sample. To eliminate variability arising from context, all the questions were presented as algorithms in vertical format. It is well established that the method a person will use to solve a problem and the success they have is easily affected by elements of a real world setting (Carraher, Carraher and Schliemann, 1987).

Students were asked to show their working and to explain in writing the methods they used. Most methods could be identified from this information. When method could not be categorised, the student was interviewed. For example, in writing some students had only described their methods as: "Did it in my head", so all of these students were interviewed. Each student was classified as predominantly using taught or invented methods. Taught methods included decomposition (the method taught at the school) and equal addition and minor variants of both. A brief explanation of each method appears in Figure 1. One example of a minor variant is when a student who is calculating in the units column of the example in Figure 1 does not subtract 9 from 16, but subtracts 9 from 10 and then adds 6. This variation of both decomposition and equal additions is sometimes taught, because only subtraction facts from 10 are used.

Figure 2: Test items for subtraction

52	246	203	1004
-27	-159	-65	-568

Mathematics Achievement

Students from Grade 8 to 12 were classified as "above average", "average" and "below average" in mathematical ability Two different ways of classifying were used: for Grades 8 to 10 the mathematics mark which the student obtained in the common first semester exam was used, and for Grades 11 and 12 the particular class the student was in was used. Here class allocation is on the basis of previous examination marks and level of mathematics studied. Grade 7 students were not classified by achievement.

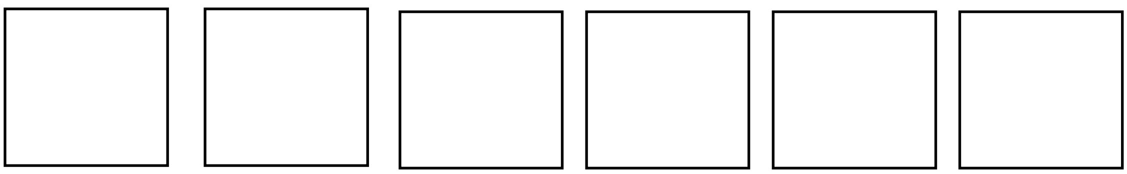
Lateral Thinking Ability

A three item assessment of lateral thinking was created and piloted on a small sample. The first two items were adapted from de Bono (1970) and the third item was adapted from Entwistle and Ramsden (1983), who used it as an item for "divergent thinking", which they claim is closely related to lateral thinking. In its final form (see Figure 3) the assessment was administered to all 436 of the boys in Grades 11 and 12 who were present on a specified day. This was 86% of the total enrolment. No follow up of absent students was possible on this occasion. Grades 11 and 12 only were assessed because these were the grades that had large numbers of students using invented methods. The items were scored by giving one point for each different answer to each question and half a point for answers that were clearly derivatives of other answers or not adequately explained. Raw scores ranged from 4 to 26.

Figure 3: Assessment of lateral thinking ability

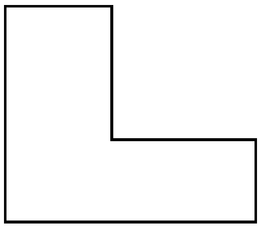
This is a short test to see how you think in a particular way. You have ten minutes to complete it.

Q.1 You are to consider the squares below. For each square, you are asked to divide it into four equal pieces, using a different method each time. If you can think of more than six different ways, draw more squares on the back of the page and fill them in. Equal pieces means pieces of equal shape.

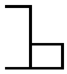


Q.2 Write down as many uses as you can think of for a leather belt.

Q.3 Consider the following figure:



You are to describe it in as many ways as you can. Two ways have been done for you.
(Draw any mathematical shapes you suggest, as in the example.)

(1) Two rectangles joined together. 
(2) The letter L.

Results

Frequency of Use of Taught and Invented Methods.

The numbers and percentages of students using taught and invented methods are shown in Table 1. The percentage of students using invented methods rises steeply from almost no students at Grade 7 to over 25% at Grade 12. This is shown graphically in Figure 4. A correlation coefficient of 0.96 between grade level and the logarithm of the proportion of students using an invented method confirms that the rise is exponential.

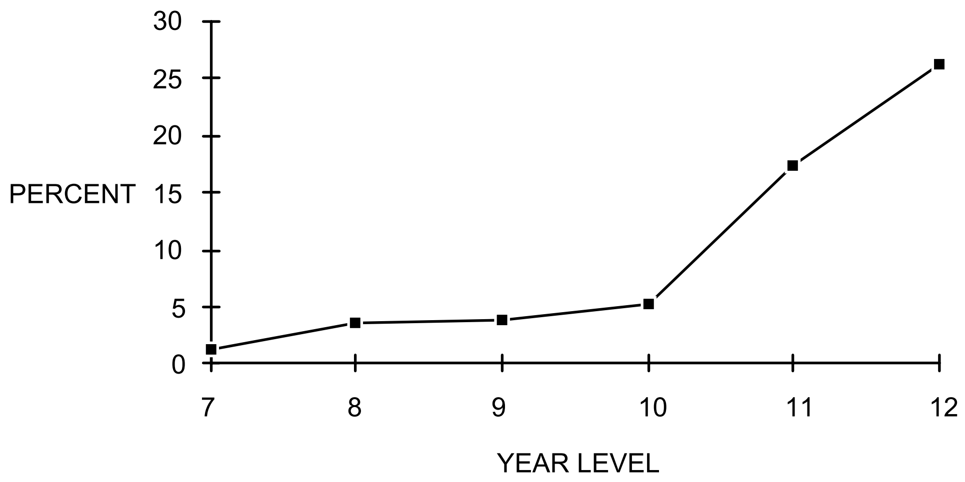
Table 1

Numbers of students in each grade using taught and invented methods. (Percentages of grade given in brackets.) (N= 1362*)

		Subtraction method	
Grade	n	taught	invented
7	223	220 (98.7%)	3 (1.3%)
8	191	184 (96.3%)	7 (3.7%)
9	252	242 (96.0%)	10 (4.0%)
10	209	198 (94.7%)	11 (5.3%)
11	259	214 (82.6%)	45 (17.4%)
12	228	168 (73.7%)	60 (26.3%)

* Eight students were not included in this table since their methods could not be categorised as taught or invented.

Figure 4. Percentage of students in each grade using invented methods.



Because this is a cross-sectional rather than a longitudinal study, in order to infer that students become more likely to use invented methods as they become older, it must be known that the population is constant. In fact, this is the case. The school population is quite stable from year to year, with very few boys leaving to attend other schools. There were intakes of 30 or fewer students per annum from other schools into all Grades from 8 to 11. The subtraction methods of new arrivals to the school were examined and it was found that amongst the new arrivals the proportions of boys using invented and taught methods were generally similar to the proportions in the pupils continuing at the school at each level. Details are given in Perry (1991). Staff in the secondary school and the associated elementary school confirmed there had been no known change in the teaching of subtraction whilst the present students had been at school. Decomposition had been taught for many years. For these reasons, the increase in the use of invented methods of subtraction cannot be attributed to changes in the school population, but is most likely caused by boys changing from using taught methods to invented methods as they grow older.

A wide variety of invented methods were used in all questions. Some of the methods used for the first question (52 - 27), for example, were

- rounding off (such as 52 - 27 = 50 - 30 + 2 + 3 = 25)
- complementary addition (27 + 3 gives 30, add 20 to give 50, add 2 to give 52)
- step subtraction (such as 52 - 20 - 7= 25)
- subtracting a common number from both (such as 52 - 27 = (52 - 2) - (27 - 2))
- using negatives as was illustrated at the beginning of this paper.

Students did not consistently use one invented method, but confidently swapped from question to question, as is shown by the following transcript from an interview with Donald in the pilot study.

Interviewer. Try this question: 52 - 27.

Donald. Add 20 to 27, which gives 47. Then add 5 to get 52. So the answer is 25.

Interviewer. Now try 246 - 159.

Donald. First 159 plus 100 is 259. Too much! But 259 - 246 gives 13 so 100 - 13 is 87 and the answer is 87.

Association between Achievement and Use of Invented Methods.

Table 2 shows the numbers of students in the three categories of mathematical achievement (classified as described above) using taught or invented algorithms, for combined Grades 8 to 12. Students in the below average achievement group were more than twice as likely to use invented methods as were students in the above average group. A Chi-square test confirmed that the differences in proportions were statistically significant. (χ^2 = 9.23, d.f. = 2, p = 0.01). The same result held at most of the individual grade levels.

Table 2

Numbers of students using taught and invented methods in each mathematical achievement group. (Percentages of achievement group given in brackets) (N = 973)

		Subtraction method	
Achievement	n	taught	invented
Above average	232	209 (90.1%)	23 (9.9%)
Average	587	515 (86.0%)	72 (14.0%)
Below average	154	123 (79.9%)	31 (20.1%)

Since some of the invented methods used seemed to us to demonstrate excellent understanding of subtraction and place value and were often quite complicated, it seemed possible that a significantly higher proportion of the very high achievers may have used invented methods. However, a further Chi-square test showed that this was not so. Details of these further statistical tests are given in Perry (1991).

Association between Accuracy and Use of Invented Methods.

In total, 87 students (6.4%) obtained one or more incorrect answers to the four subtraction questions. The proportion of students making errors decreased steadily with grade level, from 11.2% at Grade 7 to 3.5% at Grade 12. This trend was found to be significant using a Chi-square test of trend in proportions (χ^2 = 14.93, d.f. = 1, p < 0.01)

Table 3 shows the numbers of students who made and did not make errors classified by method of subtraction. Clearly, there is no association of errors with use of taught or invented methods. Students using taught methods were as likely to be correct as students using invented methods. This is so, despite the fact that below average achievement students were over-represented among the users of invented methods.

Table 3

Number of students making and not making errors using taught and invented methods. (Percentages of each in the error category in brackets) (N= 1362)*

		Subtraction method	
Error category	n	taught	invented
No errors	1283	1155 (90.0%)	128 (10.0%)
One or more errors	79	71 (89.9%)	8 (10.1%)

* Eight students were not included in this table since their methods could not be categorised as taught or invented.

Association between Lateral Thinking and Use of Invented Methods.

The lateral thinking test was given to 436 students in Grades 11 and 12. The mean score on this test of the 343 students who used taught methods was 13.21 (standard deviation = 4.03). The mean score of the 93 students who used invented methods was 13.11 (standard deviation = 3.76). It was therefore concluded that there is no difference in the lateral thinking scores of students using taught and invented methods of subtraction. Further investigation of special subgroups failed to find any association between use of invented methods and score on the lateral thinking test. For example, students who used two or more invented methods did not score more highly on the lateral thinking test than others. There was no tendency for students who scored very highly on the lateral thinking test to use more invented methods or conversely, for students with very low scores to use more taught methods. Details of this further analysis are given in Perry (1991). In summary, use of invented methods seems unrelated to the measure of lateral thinking ability tested here.

Discussion

This study has documented the incidence of use of taught subtraction algorithms as students progress through secondary school. As expected accuracy in subtracting increases with age. Interestingly, lower achieving students tend to use more invented methods than do others, but there is no association between use of invented methods and number of errors. There is a steady increase in the proportion of students using an invented method from Grade 7 to 12. In fact the proportion rises exponentially over this time. The switch to invented methods seems particularly pronounced at Grade 11, even though all students are still studying mathematics at school at that time. Discussions with the teachers did not uncover any reason for the increased use of invented methods that could be related to curriculum or teaching practices. Calculators are permitted in all mathematics classes from Grade 7 upward.

The most likely explanation for the increase in the use of invented methods is that students forget the taught algorithms through lack of use. This explanation is supported by the data that indicated that the below average students were more likely to use invented methods and that the use of invented methods did not seem to be associated in any way with creative, lateral thinking skills. As they get older, students confidently begin to employ their own understanding to invent methods, rather than rely on (partly?) forgotten algorithms. Some of the interviews with students conducted as part of the pilot study for this work also supported this conclusion.

Figure 5. Two categories of methods used to calculate 246 - 159.

Rounding Off

250 - 160 - 4 + 1		
250 - 150 - 4 - 9		
246 - 200 = 46,	200 - 159 = 41,	46 + 41 = 87
246 - 160 + 1		
246 - 146 = 100,	159 - 146 = 13,	100 - 13 = 87
246 - 100 = 146,	150 - 60 = 90,	90 - 4 - 1 + 2 = 87
245 - 160 + 1 + 1		
240 - 160 + 1 + 6		
240 - 150 + 6 - 9		
240 - 100 - 50 + 6 - 9		
200 - 160 = 40,	40 + 40 + 6 +1 = 87	
200 - 113		

200 - 100 = 100, 59 - 46 = 13, 100 - 13 = 87
200 - 100 = 100, 46 - 59 = -13, 100 - 13 = 87

Addition
159 + 1 + 40 + 40 + 6 = 246 Ans 87
159 + 1 + 80 + 6 = 246 Ans 87
159 + 41 + 46 = 246 Ans 87
159 + 7 + 80 = 246 Ans 87
159 + 100 = 259, 259 - 13 = 246, 100 - 13 = 87
159 + 90 = 249, 249 - 3 = 246, 90 - 3 = 87
159 + 80 + 7 = 246 Ans 87
160 + 86 + 1 = 246 Ans 87
159,169,179,189,199,219,229,239 (eight tens), 239 + 7 = 246
80+50=130, 7 + 9 = 16, 87 + 16 = 146 Ans 87.

The variety of invented methods used by students is illustrated in Figure 5 by some of the methods used to calculate 246 - 159. Methods used were classified as "Subtraction", carried out in 7 ways, "Rounding Off" (14 ways), "Addition" (10 ways), and there were 6 miscellaneous ways used by students in the sample. As far as can be ascertained, each part of an answer shown in Figure 5 was carried out as a separate step. So, for example, if a student usign a subtraction method who used the first method given in Figure 5 (indicated as 246 - 150 - 9) seemed to have subtracted 150 in a single step , to give 96 and then subtracted 9, giving 87 in a single step.

Ideally, the assessment of lateral thinking would have been more rigorously developed before use. No standard test was available. However, the results of the assessment showed almost no difference between the two groups. Hence further effort in refining the test to identify differences between the users of invented and standard methods would not be warranted. Creative, lateral thinking has to do with responses in new situations. We propose that students see the subtraction questions as routine.

As shown earlier and given in more detail by Perry (1992), great variety and creativity was exhibited in the invented methods used. No doubt that if the study was repeated on a different group of students, or if you, the reader, answered the questions, new methods would appear. It is important to note that for most students, invented methods are *ad hoc*, changing according to the question (many students used more than one invented method in answering the four items) and probably also simply from time to time. Three of the six students who had used invented methods in the pilot study consistently used decomposition in the main study two months later. It seems that invented methods generally do not attain for their users the status or stability of routine algorithms.

The study has gone some way towards establishing the place of formal algorithms in the calculation procedures that people use. In this school setting, with questions presented in a "school" way, the vast majority of students used taught algorithms, generally correctly. The use of invented methods seems to be a response to forgetting, rather than a rejection of the taught algorithm. However, by the end of secondary schooling, a quarter of the students demonstrated a preference for their own methods. This suggests that a substantial proportion of students do not find the algorithms satisfactory. As has been noted from other studies, in a real life context or where pencil and paper were not the primary aids available, the swing away from school methods could be expected to be much greater. It might also be expected that in the general school population the swing to invented methods would be more pronounced than in this study, because this data comes from a school with higher than average academic standards as evidenced, for example, by the low error rate in the four subtraction questions.

In our opinion, these findings support changes advocated by the NCTM *Standards* (1989) and given theoretical support by a constructivist epistemology, to de-emphasise the teaching of standard algorithms and to offer positive encouragement to students to invent and discuss their own methods. After students leave school, basic understanding of subtraction is more likely to be retained than rules of a standard algorithm.

References

Australian Education Council. (1991). *A National Statement on Mathematics for Australian Schools*. Carlton, Vic.: Curriculum Corporation.

Backman, C. A. (1978). Analysing children's work practices. In M. N. Suydam, & R. E. Reys (Eds.). *Developing computational skills*. (pp. 177-195). Reston, VA: National Council of Teachers of Mathematics.

Bell, A. W., Costello, J. & Kuchemann, D. E. (1983). *A review of research in mathematical education. Part A Research on learning and teaching.* Windsor: NFER.

Carraher, T. N. & Schliemann, A. D. (1985). Computational routines prescribed by schools: Help or hindrance? *Journal for Research in Mathematical Education,* 16(1), 37-54.

Carraher, T. N., Carraher, D. W., & Schliemann ,A. D. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education,* 18(2), 83-97.

Davies, H. B. (1978). A seven-year-old's subtraction technique. *Mathematics Teaching*, 83, 15-16.

Davis, R. B. (1984). *Learning mathematics. The cognitive science approach to mathematics education.* London: Croom Helm.

de Bono, E. (1970). *Lateral thinking. A textbook of creativity.* London: Ward Lock Educational.

Entwistle, N. J., & Ramsden, P. (1983). *Understanding student learning*. London: Croom Helm.

Ginsberg, H.P. (1977). *Children's arithmetic: the learning process*. New York: Van Nostrand.

Jones, D. A. (1973). Don't just mark the answer - have a look at the method! *Mathematics in School*, *4*(3), 29-35.

Labinowicz, E. D. (1985). *Learning From children: New beginnings for teaching numerical thinking*. Menlo Park, California: Addison-Wesley.

Marks, J. L., Hiatt A. A., & Neufeld, E. M. (1985). *Teaching Elementary School Maths for Understanding. 5th Edition*. New York: McGraw Hill.

Maher, C.A. & Alston, A. (1990). Teacher development in mathematics in a constructivist framework. In R. B. Davis, C. A. Maher and N. Noddings (Eds.), *Constructivist Views on the teaching and learning of mathematics.* (pp.147 - 165). Reston, VA: National Council of Teachers of Mathematics.

Matthews, J. (1981). An investigation into subtraction. *Educational Studies in Mathematics*, 12, 327-338.

McIntosh, A. (1978) .Some subtractions: What do you think you are doing? *Mathematics Teaching*, 83, 17-19.

Ministry of Education, Victoria (1988). *The mathematics framework P-10*. Melbourne: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics* Reston, VA: Author.

Perry, A. D. (1991). *Subtraction methods of secondary school students.* Unpublished M.Ed. thesis, University of Melbourne, Australia.

Perry, A. D. (1992). Subtraction Methods of Secondary School Students. *Australian Mathematics Teacher*, 48(4), 24-27.

Petitto, A. L., & Ginsburg, H. P. (1982). Mental arithmetic in Africa and America: Strategies, principles and explanations. *International Journal of Psychology*, *17*, 81-102.

Plunkett, S. (1981). Decomposition and all that rot. In A. Floyd (Ed.), *Developing mathematical thinking* (pp. 177-185). London: Addison Wesley.

Trafton, P. R. (1978). Estimation and mental arithmetic; important components of computation. In M. N. Suydam, & Reys, R. E. (Eds.), *Developing Computational Skills.* (pp. 196-213). Reston, VA: National Council of Teachers of Mathematics.