TEACHING NEGATIVE NUMBER USING INTEGER TILES

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The paper reports some of the findings of a study involving use of an integer tiles embodiment of the neutralisation model. Short and long-term learning outcomes of negative number teaching using integer tiles and normal number-line based teaching methods are compared.

Need for a New Approach

Negative number is widely regarded as a difficult topic; teachers consider it hard to teach and many students experience learning difficulties. In particular use of the number-line as the introductory model for the teaching of integer operations appears to be providing little understanding for many students. Although integer addition is relatively easy to illustrate on a number-line using directed line segments, explanations for subtraction tend to be clumsy and confusing. Some textbooks (e.g. Daly et al., 1993; Lynch et al., 1991) give 'facing and pacing' explanations which cause problems including mixing the meanings assigned to the operation (minus) and the integer sign (negative). The fact that subtraction is sometimes interpreted as 'take away' and sometimes as 'difference' is another source of confusion.

Lack of understanding usually results in students relying on rote-learning of sign rules which frequently become misapplied across the operations. Our testing of students in Years 9, 10 and 11 revealed that more students, at each year level, gave wrong answers to the addition of two negatives than for the product of two negatives (Figure.1). (No prize for identifying their error!)

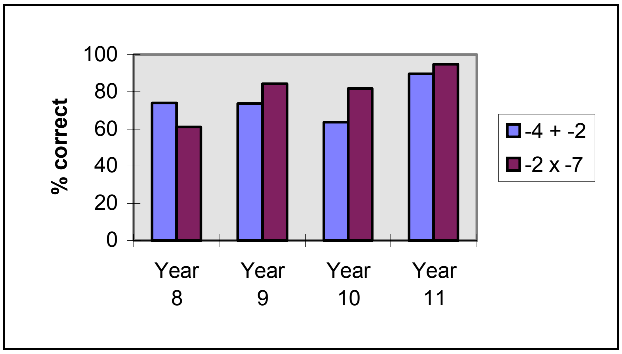

Figure 1.1 Comparison of percentages of students in Years 8, 9, 10 & 11 correct on sum and product of negatives items. (N = 23 + 19 + 22 + 39 = 103)

A preliminary investigation, preceding the experimental comparative study (Hayes, 1998) outlined in this paper, revealed that less than half of the Year 8 class tested, five months after negative number was initially taught, were competent in integer subtraction and only slightly over half showed competency in multiplication and division. In the Year 9 class tested only around 60% of the students were able to perform any given small integer subtraction, multiplication or division. The Year 10 class performed no better.

In Year 11 students get the chance to drop mathematics. Thus it could be expected that students electing to study mathematics would have attained mastery of basic integer processes. This was not the case. Only one item $(8 + {}^{-}8)$ resulted in all students getting the correct answer. Some consistently got the subtraction items wrong and, as mentioned earlier, could not manage ${}^{-}4 + {}^{-}2$. There were also errors on multiplication and division items. Thus it appears that failure to attain understanding and mastery of basic integer operations at the time of initial teaching produces persistent weaknesses that some find very difficult to overcome. This illustrates the need for more effective teaching at junior secondary level.

Also evident was an appalling level of basic algebra skills in items used for the purpose of testing integer skills in an algebraic context. Only around a quarter of students in Years 9 and 10 could cope with expanding and simplifying $8 - 4(x - 2)$. Twenty six out of the 39 students in Year 11 could not perform the same task. The preliminary investigation provided clear justification for the need to develop a fresh approach to the teaching of negative number.

The Experimental Approach

The experimental teaching method developed for the purpose of the study used sets of small (2 cm. square) double-sided *tiles* which will be referred to as *integer tiles*. It is a development of the neutralisation model using two colours of counters. Freudenthal (1983) calls it the annihilation model. Despite appearing in some form or another from time to time in journal articles (e.g. Haner, 1947; Rowland, 1982; Battista, 1983; Hollingsworth, 1992; Lytle, 1994) and occasionally in textbooks (e.g. Del Grande et al., 1973; Blane & Booth, 1991) the model is seldom used in secondary-level classrooms. The fact that it is a structured aids approach requiring materials preparation, organisation and handling may deter some teachers. Lack of knowledge and understanding of the model and lack of in-depth attempts at classroom research and evaluation of methods based on the model are other possible reasons for its neglect.

Two kinds of integer tile were used; one labelled '${}^{+}1$' and '${}^{-}1$' on opposite faces and the other labelled '0' on both faces (Figure 1).

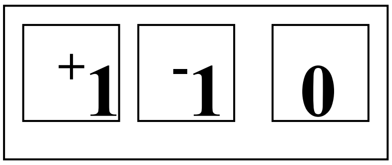

Figure 1. Integer tile faces

The intention was to contrast learning and operational skill outcomes for (experimental) classes using the integer tiles with outcomes for comparable ability (control) classes taught using number-line based methods in conjunction with normal textbooks and teaching. Two work booklets were written by one of the authors specifically for use with the integer tiles.

Experimental Design

Three Victorian eastern suburban coeducational secondary schools (two government, one private) were involved in the study. The government schools (A & B) begin teaching negative number early in Year 8. The private school (C) introduces the topic in term three of Year 7. The main teaching and testing program began in 1995. The final test was given in mid-1997. The preliminary investigation referred to previously was conducted in School B in 1994. Pilot studies were conducted in Schools A and C during 1994 for the purpose of trialing the experimental materials (tiles and work booklets) and tests and familiarising teachers with the neutralisation model.

Teachers and classes

In each of the schools there were experimental and control classes. The students in the experimental classes were taught by their usual mathematics teachers who were willing to adopt the experimental approach. Students worked, substantially at their own pace, exclusively from and in the especially designed work booklets. Teachers organised their classes and provided explanations and assistance when required. Control classes continued with the normal methods adopted by their teachers. In each of the control classes the teaching was generally based on the approach contained in the textbook chosen by each school.[1] In School B the same teacher taught both the experimental and control classes. The teachers in Schools A and C taught either an experimental or a control class.

In each school it was necessary to use existing classes as time-tabling constraints made random allocation of students to teaching methods impractical. However the classes involved were all non-streamed and considered by their teachers to be of approximately comparable ability ranges.

Testing and teaching program

The students, in each class involved, were pre-tested in the lesson immediately prior to beginning the three weeks of lessons (one per day) devoted to the teaching of the topic. Classes were post-tested in the mathematics lesson after completing the topic. The first retention test was administered approximately six months later. A long term retention test was given to as many as possible of the 1995 cohort in mid 1997.

Table 1 summarise the details and sequence of the testing and teaching activities for each of the classes. The numbers of students (N) shown were the numbers on the class lists. Some students were not present for all tests. Pseudonyms have been given to the teachers. (The asterisks indicate that all classes received each test.)

Table 1
Testing and Teaching Activities

School	Year	Class	Teacher	N	Test W	Teaching	Test X	Test Y	Test Z
A	8	Expt.	Helen	19	*	Tiles	*	*	*
	8	Cont.	Irene	19	*	Normal	*	*	*
B	8	Expt.	Jack	25	*	Tiles	*	*	*
	8	Cont.	Jack	24	*	Normal	*	*	*
C	7	Expt.	Kath	19	*	Tiles	*	*	*
	7	Expt.	Laurie	20	*	Tiles	*	*	*
	7	Cont.	Michelle	14	*	Normal	*	*	*

Test W = Pre-test Test X = Post-test Test Y = Retention test
Test Z = Long term retention test

[1] Schools A and B used *Mathematics Today Year 8* (Daly et al.). School C used *Maths 8* (Lynch et al.).

Pre-test

Test W covered knowledge of basic positive number facts (addition and multiplication) and also included 14 simple positive and negative integer operation items. The latter items were intended as indicators of entry level knowledge and intuitions with regard to integer operations.

Post-test

Test X contained ten questions covering; a range of integer operations (involving completing equations, compiling tables and evaluating expressions), word problems (themes were temperatures and altitudes) and order.

Retention test

Test Y consisted of three questions. Question one was basically identical to that used as question one in Test X to facilitate direct performance comparisons for particular operational skills. Only the numerical values were changed. Question two tested knowledge and application of the sign rules in the context of binary operations between pairs of large integers. The items in question three involved making integer substitutions in and then evaluating a range of simple algebraic expressions.

No attempt was made by the researcher to influence any teaching relating to the use of integers and negative number during the interim period. It was expected that all students would be applying negative numbers and their extended knowledge of the real number system to other topics that had followed (e.g. solving equations, algebraic manipulations, substitutions in and evaluation of formulae and algebraic expressions, and graphing).

Long-term retention test

In Test Z the questions were presented in an algebraic form focussing on negative number skills required for continuing in mathematics including; substitution in and evaluation of simple algebraic expressions, use of sign rules and ability to make bracket expansions, simplifications requiring collection of like terms and simple formulae evaluations. A word problem involving scoring positive and negative points in a quiz show was also included.

Results

The overall test results showed that;

- the experimental and groups were closely matched on the pre-test (Table 2),
- the experimental group performed significantly better than the control group on both the post-test and the first retention test (Tables 3 & 4),
- the performances of the groups converged on the long-term retention test two years later and the difference in favour of the experimental group was not significant (Table 5).

In addition to the general group comparisons, shown in the tables, detailed across and within schools item analyses and comparisons were made for each of the tests. Extensive qualitative data from observations, conversations and interviews with teachers and students and examination of students' work booklets and workbooks were also gathered during the course of the study.

Table 2
Test W: Group mean scores and standard deviations

Group	Positive number facts (/30)		Integer operations (/14)		Total score (%)	
	Mean	SD	Mean	SD	Mean	SD
Expt. (n = 84)	29.1	2.6	7.8	3.4	83.1	9.6
Cont. (n = 58)	28.4	2.6	7.6	3.9	82.6	11.2
t-test (t, d.f., p)					0.08, 143, 0.934 (n.s.)	

Table 3.
Test X: Group mean scores and standard deviations

Group	Integer operations (/30)		Total score (%)	
	Mean	SD	Mean	SD
Expt. (n = 82)	23.1	6.3	77.4	18.3
Cont. (n = 57)	20.0	7.8	66.5	24.2
t-test (t, d.f., p)			3.03, 137, 0.003 (sig.)	

Table 4
Test Y: Group mean scores and standard deviation

Group	Integer operations (/30)		Total score (%)	
	Mean	SD	Mean	SD
Expt. (n = 74)	24.4	6.3	79.2	21.8
Cont. (n = 58)	19.8	7.7	63.9	23.9
t-test (t, d.f., p)			3.83, 130, 0.0006 (sig.)	

Table 5.
Test Z:Group mean scores and standard deviations

Group	Integer operations (/10)		Total score (%)	
	Mean	SD	Mean	SD
Expt. (n = 56)	8.8	2.0	76.4	19.9
Cont. (n = 35)	8.3	2.2	70.3	21.6
t-test (t, d.f., p)			1.35, 89, 0.180 (n.s.)	

Test findings

Detailed analysis of the post-test questions and items revealed that the experimental group,

- produced significantly better results than the control group in eight of the ten question categories tested (results on multiplication and division were comparable),
- showed a higher proportion of students giving correct answers on 33 of the 41 items analysed in detail,
- was shown to be generally superior on integer addition and subtraction,
- performed better on items (equations, brackets, mixed operations) requiring the use of integer operations in conjunction with other mathematical knowledge (e.g. order of operation),
- performed better on the tasks of ordering negative numbers and answering word problems.

By the time of the post-test students in both groups were using sign rules for calculations. The experimental group students had been expected to discover and articulate the sign rules for themselves through the neutralisation model-based integer tile activities and associated exercises in the work booklets. The control group students were taught addition and subtraction using the number line however the sign rules tended to be given rather than thoroughly explored. A consequence of the contrasting approaches was that a larger proportion of experimental class students seemed to have developed a more confident and secure knowledge of the rules and showed less tendency to confuse sign rules across operations. This appeared to be the main reason for the difference in performance levels observed on the post-test.

On the retention test the experimental group showed substantially higher proportions of students correct on all of the 62 items. In the six month period between the post-test and the retention test the skill gap on integer operations appeared to widen between the groups.

The only mathematical program difference between the groups was during the three-week negative number teaching experiment. The final worksheet in Work Booklet Two contained a short set of substitution and evaluation exercises. The experimental class teachers treated it as an optional extra because the topic of symbolic algebra[2] (linear algebra[3]) had not been taught. By the time of Test Y all classes had been taught the latter topics based on the approaches contained in the chosen textbooks.

The long-term retention test revealed that the performance gap found around 18 months earlier in favour of the experimental group, had substantially closed. In general the former experimental class students seem to have continued at about the same measured knowledge and skill levels, whilst the former control students had markedly improved. It seems that the teaching received by the former experimental class members did not build on and effectively develop their superior early negative number skills.

Most experimental class students became competent at integer calculations although some showed subtraction weaknesses. The evidence suggests that subtraction is the most difficult negative number calculating skill. Compared with the number line method used, the integer tile method was instrumental in developing much higher levels of subtraction skill in the early part of the study. However some experimental students did not master it at the time of teaching and weaknesses had been allowed to persist.

Since the initial teaching in 1995, the students in each school were regrouped at least twice and taught by different mathematics teachers. No attempt was made by the researchers to influence teaching methods. Regardless of the initial negative number teaching method used students invariably end up relying on the operational sign rules. The tile-based activities encouraged experimental class students to find, understand and articulate the rules themselves during the initial three-week teaching program. However the impact of earlier teaching diminishes and some students forget. Mathematics teachers thus find it necessary to re-teach or revise topics to classes or individuals.

With regard to negative number, the researcher's observations of classroom events indicate that many teachers shortcut the teaching process and merely re-state the sign rules (often in carelessly abbreviated form) without appropriate justification and/or reference to a suitable model. As a consequence uncertainty and confusion with rules

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[2] Lynch et al. Chapter 4.

[3] Daly et al. Unit 12.

can result for weaker students. This is almost certainly a causal factor in the lack of mastery observed in some students.

Weakness in basic algebra prevented many students (both experimental and control) from demonstrating signed number skills in the context of algebra. The suggestion offered is that algebra should be taught using only the numbers of arithmetic over a reasonable period (perhaps a year) before complicating it with the introduction of negative number.

Classroom Observations and Impressions

Whilst actively studying the topic most experimental class students demonstrated a good understanding of the neutralisation model using the integer tiles. The students were actually engaged in structured integer experiments. This appeared to facilitate acceptance of the operational rules through the opportunity and encouragement provided for finding and articulating the rules for themselves. As much as possible the rules were kept in the background and students appeared to be calculating successfully independently of stated rules. By the end of the three week unit the students were expected to have moved beyond the tiles and were using the sign rules. In contrast the textbook based teaching provided the control class students with little opportunity for genuine exploratory activity. The manner in which the number line was used provided little evidence of understanding of operations in any of the three control classes. The main classroom activity seemed to be training in using the sign rules.

For weaker students in particular, the three weeks teaching time given to the topic of integers (or directed number) is insufficient to develop embedded understanding. Furthermore reinforcement of negative number skills is not taking place in the context of following application topics. Periodic revision of negative number operations and related interpretational skills is necessary. Perhaps the occasional use of proofs, using integer tiles to illustrate the sequence of steps, may be useful occasional exercises in Years 9, 10, …, (e.g. Prove that $1 - {}^{-}4 = 5$. Show that ${}^{-}4 \times {}^{-}3 = 12$.) This type of activity tackled in a discursive and exploratory manner with a class, small group or individual may produce better learning than that resulting from simply getting students to do sets of meaningless 'training in sign rules exercises'. It would also give students opportunity to talk about the expressions and help develop their language and interpretation skills as advocated by Mukhopadhyay (1995).

Classroom observations and conversations with teachers and students suggest that the integer tile embodiment of the neutralisation model provides a more effective initial negative number teaching instrument than the number line for the following reasons;

- it facilitates clear explanations and demonstrations of all integer operations except the product of a negative by a positive and the product of two negatives[4] and division of a positive by a negative. (Integer tile subtraction and the distributive rule was used for developing the two awkward products.)
- the materials promote dialogue and cooperation between students and assist the development and use of language skills in the context of mathematics because the students find the operations easy to describe and the tiles easy to manipulate,

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[4] Use could be made of the reversible property of the tiles for mixed sign multiplications. This was not attempted in this study but the idea merits further thought and investigation. Del Grande et al. use the idea in their *Math Book 2*.

- the operations are dynamically linked with natural actions and common usage of words (i.e. addition as combining, subtraction as take-away),
- the zero tile makes the neutralisation and neutral pair forming processes easier to demonstrate, explain and understand, countering to some extent a criticism of the model due to the difficulty that some students find with such, according to some critics, 'unnatural' processes,
- the operations are easy to visualise and this can be encouraged by using large numerical values among the exercises to make use of the imagination essential,
- tile manipulations can be translated into written calculations and this was encouraged by guided activities in the work booklets. However although most students coped with such tasks, they seemed less able and less inclined to construct chains of written calculations for themselves. In general students considered such a process to be unnecessary.

Conclusion

The data collected in this study suggests that the integer tile embodiment of the neutralisation model appears to be superior to the number line model, as currently interpreted and used in popular textbooks, for the purpose of initial teaching of negative number operations and justifying sign rules. However attention needs to be given to the problem of maximising the benefits of the improved acquisition of operational skills facilitated by the model during the teaching of subsequent application topics.

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